JEE(ADVANCED) 2013

Paper - 1 [Code - 5]

MATHEMATICS

SECTION – 1 : (Only one option correct Type)

This section contains 10 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which** ONLY ONE is correct.

41. Perpendiculars are drawn from points on the line $\frac{x+2}{2}=\frac{y+1}{-1}=\frac{z}{3}$ to the plane x + y + z = 3. The feet of perpendiculars lie on the line

(A) $\frac{x}{5}=\frac{y-1}{8}=\frac{z-2}{-13}$
(B) $\frac{x}{2}=\frac{y-1}{3}=\frac{z-2}{-5}$
(C) $\frac{x}{4}=\frac{y-1}{3}=\frac{z-2}{-7}$
(D) $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

Sol. **(D)**

Any point B on line is $(2\lambda - 2, -\lambda - 1, 3\lambda)$

Point B lies on the plane for some λ

$\Rightarrow (2\lambda - 2) + (-\lambda - 1) + 3\lambda = 3$

$\Rightarrow 4\lambda = 6 \Rightarrow \lambda = \frac{3}{2} \Rightarrow B \equiv \left(1, \frac{-5}{2}, \frac{9}{2}\right)$

The foot of the perpendicular from point (– 2, – 1, 0) on the plane is the point A (0, 1, 2)

$\Rightarrow$ D.R. of AB = $\left(1, \frac{-7}{2}, \frac{5}{2}\right) \equiv (2, -7, 5)$

Hence $\frac{x}{2}=\frac{y-1}{-7}=\frac{z-2}{5}$

*42. For a > b > c > 0, the distance between (1, 1) and the point of intersection of the lines ax + by + c = 0 and bx + ay + c = 0 is less than $2\sqrt{2}$, then

(A) a + b – c > 0
(B) a – b + c < 0
(C) a – b + c > 0
(D) a + b – c < 0

Sol. **(A)**

For point of intersection $(a-b)x_1 = (a-b)y_1$

$\Rightarrow$ point lie on line y = x ….. (1)

Let point is (r, r)

$\sqrt{(r-1)^2+(r-1)^2} < 2\sqrt{2}$

$\sqrt{2}\,|(r-1)| < 2\sqrt{2}$

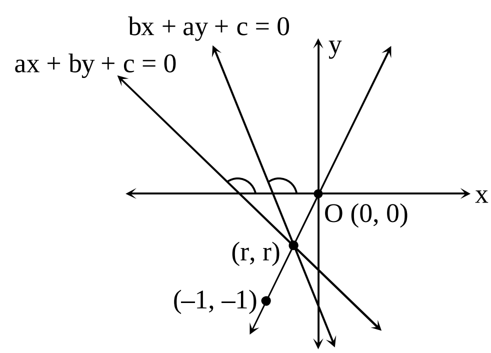
$\Rightarrow |r-1| < 2$

$-1 < r < 3$

$\Rightarrow$ (–1, –1) lies on the opposite side of origin for both lines

$\Rightarrow -a - b + c < 0$

$\Rightarrow a + b - c > 0$



43. The area enclosed by the curves y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0, \frac{\pi}{2}\right]$ is

(A) $4\left(\sqrt{2}-1\right)$
(B) $2\sqrt{2}\left(\sqrt{2}-1\right)$
(C) $2\left(\sqrt{2}+1\right)$
(D) $2\sqrt{2}\left(\sqrt{2}+1\right)$

Sol. **(B)**

$y_1 = \sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$

$y_2 = \sqrt{2}\left|\sin\left(\frac{\pi}{4}-x\right)\right|$

$\Rightarrow$ Area $= \int_0^{\frac{\pi}{4}}\left((\sin x+\cos x)-(\cos x-\sin x)\right)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left((\sin x+\cos x)-(\sin x-\cos x)\right)dx$

$= 4-2\sqrt{2}$

44. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$

(C) $\frac{3}{256}$ (D) $\frac{253}{256}$

Sol. **(A)**

P(at least one of them solves correctly) = 1 – P(none of them solves correctly)

$= 1-\left(\frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}\times\frac{7}{8}\right) = \frac{235}{256}$

*45. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0 = x_0+iy_0$ satisfies the equation $2|z_0|^2 = r^2+2$, then $|\alpha| =$

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

Sol. **(C)**

$OB = |\alpha|$

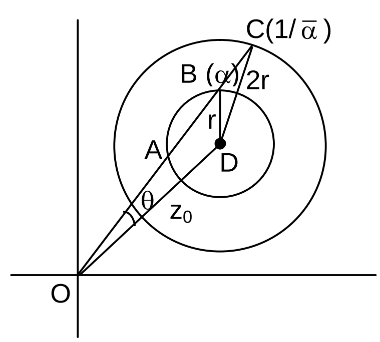
$OC = \frac{1}{|\bar{\alpha}|} = \frac{1}{|\alpha|}$

In ΔOBD

$\cos\theta = \frac{|z_0|^2+|\alpha|^2-r^2}{2|z_0||\alpha|}$

In ΔOCD

$\cos\theta = \frac{|z_0|^2+\frac{1}{|\alpha|^2}-4r^2}{2|z_0|\frac{1}{|\alpha|}}$



$$\frac{|z_0|^2+|\alpha|^2-r^2}{2|z_0||\alpha|}=\frac{|z_0|^2+\frac{1}{|\alpha|^2}-4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

46. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is

(A) 6 (B) 4

(C) 2 (D) 0

Sol. **(C)**

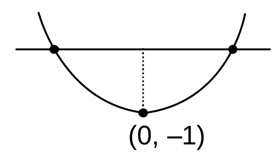
Let $f(x) = x^2 - x \sin x - \cos x \Rightarrow f'(x) = 2x - x \cos x$

$\lim_{x\to\infty} f(x) \to \infty$

$\lim_{x\to-\infty} f(x) \to \infty$

$f(0) = -1$

Hence 2 solutions.



47. Let $f:\left[\frac{1}{2},1\right] \to \mathrm{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)\,dx$ lies in the interval

(A) $(2e - 1, 2e)$ (B) $(e - 1, 2e - 1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

Sol. **(D)**

Given $f'(x) - 2f(x) < 0$

$\Rightarrow f(x) < ce^{2x}$

Put $x = \frac{1}{2} \Rightarrow c > \frac{1}{e}$.

Hence $f(x) < e^{2x-1}$.

$\Rightarrow 0 < \int_{1/2}^{1} f(x)\,dx < \int_{1/2}^{1} e^{2x-1}dx$

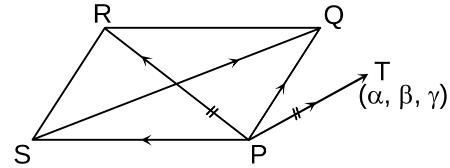
$0 < \int_{1/2}^{1} f(x)\,dx < \frac{e-1}{2}$.

48. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ *and* $\overrightarrow{PS}$ is

(A) 5 (B) 20

(C) 10 (D) 30

Sol. **(C)**



Area of base (PQRS) = $\frac{1}{2}\left|\overrightarrow{PR}\times\overline{SQ}\right| = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & -4\end{vmatrix}$

$= \frac{1}{2}\left|-10\hat{i}+10\hat{j}-10\hat{k}\right| = 5\left|\hat{i}-\hat{j}+\hat{k}\right| = 5\sqrt{3}$

Height = proj. of PT on $\hat{i}-\hat{j}+\hat{k} = \left|\frac{1-2+3}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$

Volume = $\left(5\sqrt{3}\right)\left(\frac{2}{\sqrt{3}}\right) = 10$ cu. units

*49. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is

(A) $\frac{23}{25}$

(B) $\frac{25}{23}$

(C) $\frac{23}{24}$

(D) $\frac{24}{23}$

Sol. **(B)**

$\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(n^2+n+1\right)\right)$

$\cot\left(\sum_{n=1}^{23}\tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right)\right)$

$\Rightarrow \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right) = \frac{25}{23}$.

50. A curve passes through the point $\left(1,\frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right)$, x > 0. Then the equation of the curve is

(A) $\sin\left(\frac{y}{x}\right) = \log x+\frac{1}{2}$

(B) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x+2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x+2$

(D) $\cos\left(\frac{2y}{x}\right) = \log x+\frac{1}{2}$

Sol. **(A)**

$\frac{dy}{dx} = \frac{y}{x}+\sec\frac{y}{x}$. Let y = vx

$\Rightarrow \frac{dv}{\sec v} = \frac{dx}{x}$

$\int \cos v\,dv = \int\frac{dx}{x}$

sin v = ln x + c

$\sin\left(\frac{y}{x}\right) = \ln x + c$

The curve passes through $\left(1, \frac{\pi}{6}\right)$

$\Rightarrow \sin\left(\frac{y}{x}\right) = \ln x + \frac{1}{2}$.

SECTION – 2 : (One or more options correct Type)

This section contains 5 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which** ONE or MORE are correct.

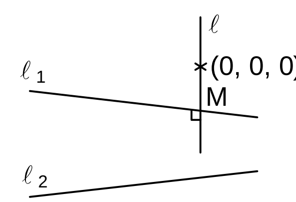
51. A line l passing through the origin is perpendicular to the lines
$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}$, $-\infty < t < \infty$, $l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}$, $-\infty < s < \infty$
Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is (are)

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B) (–1, –1, 0)

(C) (1, 1, 1) (D) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

Sol. **(B, D)**

The common perpendicular is along $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i} + 3\hat{j} - 2\hat{k}$



Let M ≡ (2λ, –3λ, 2λ)

So, $\frac{2\lambda-3}{1} = \frac{-3\lambda+1}{2} = \frac{2\lambda-4}{2} \Rightarrow \lambda = 1$

So, M ≡ (2, –3, 2)

Let the required point be P

Given that $PM = \sqrt{17}$

$\Rightarrow (3 + 2s - 2)^2 + (3 + 2s + 3)^2 + (2 + s - 2)^2 = 17$

$\Rightarrow 9s^2 + 28s + 20 = 0$

$\Rightarrow s = -2, -\frac{10}{9}$

So, P ≡ (–1, –1, 0) or $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

52. Let f(x) = xsin πx, x > 0. Then for all natural numbers n, f ′(x) vanishes at

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$ (B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(C) a unique point in the interval (n, n + 1) (D) two points in the interval (n, n + 1)

Sol. **(B, C)**

We have $f'(x) = \sin\pi x + \pi x \cos\pi x = 0$

$\Rightarrow \tan\pi x = -\pi x$

$\Rightarrow \pi x \in \left(\frac{2n+1}{2}\pi, (n+1)\pi\right) \Rightarrow x \in \left(n+\frac{1}{2}, n+1\right) \in (n, n+1)$

*53. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s)

(A) 1056 (B) 1088

(C) 1120 (D) 1332

Sol. **(A, D)**

$$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2 = \sum_{r=0}^{(n-1)} \left((4r+4)^2 + (4r+3)^2 - (4r+2)^2 - (4r+1)^2\right)$$

$$= \sum_{r=0}^{(n-1)} \left(2(8r+6) + 2(8r+4)\right)$$

$$= \sum_{r=0}^{(n-1)} (32r+20)$$

$= 16(n - 1)n + 20n$

$= 4n(4n + 1)$

$$= \begin{cases} 1056 \text{ for } n = 8 \\ 1332 \text{ for } n = 9 \end{cases}$$

54. For 3×3 matrices M and N, which of the following statement(s) is (are) NOT correct ?

(A) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj(MN) for all invertible matrices M and N

Sol. **(C, D)**

(A) $(N^TMN)^T = N^TM^TN = N^TMN$ if M is symmetric and is $-N^TMN$ if M is skew symmetric

(B) $(MN - NM)^T = N^TM^T - M^TN^T = NM - MN = -(MN - NM)$. So, (MN – NM) is skew symmetric

(C) $(MN)^T = N^TM^T = NM \neq MN$ if M and N are symmetric. So, MN is not symmetric

(D) (adj. M) (adj. N) = adj(NM) $\neq$ adj (MN).

55. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 (B) 32

(C) 45 (D) 60

Sol. **(A, C)**

Let the sides of rectangle be 15k and 8k and side of square be x then (15k – 2x)(8k – 2x)x is volume.

$v = 2(2x^3 - 23kx^2 + 60k^2x)$

$$\left.\frac{dv}{dx}\right|_{x=5} = 0$$

$$6x^2 - 46kx + 60k^2\big|_{x=5} = 0$$

$6k^2 - 23k + 15 = 0$

$k = 3,\ k = \frac{5}{6}$. Only k = 3 is permissible.

So, the sides are 45 and 24.

SECTION – 3 : (Integer value correct Type)

This section contains 5 questions**. The answer to each question is** single digit integer**, ranging from 0 to 9 (*both inclusive*).**

56. Consider the set of eight vectors $V = \left\{ a\hat{i} + b\hat{j} + c\hat{k}\,;\, a, b, c \in \{-1, 1\} \right\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is ________

Sol. **(5)**

Let (1, 1, 1), (–1, 1, 1), (1, –1, 1), (–1, –1, 1) be vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ rest of the vectors are $-\vec{a}, -\vec{b}, -\vec{c}, -\vec{d}$ and let us find the number of ways of selecting co–planar vectors.

Observe that out of any 3 coplanar vectors two will be collinear (anti parallel)

Number of ways of selecting the anti parallel pair = 4

Number of ways of selecting the third vector = 6

Total = 24

Number of non co–planar selections = ${}^8C_3 - 24 = 32 = 2^5$, p = 5

Alternate Solution:

Required value = $\dfrac{8 \times 6 \times 4}{3!}$

$\therefore$ p = 5

57. Of the three independent events E_1, E_2, and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\, p = \alpha\beta$ and $(\beta - 3\gamma)\, p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). Then $\dfrac{\text{Probability } of\ occurrence\ of\ E_1}{\text{Probability } of\ occurrence\ of\ E_3}$ = ___________

Sol. **(6)**

Let P (E_1) = x, P (E_2) = y and P (E_3) = z

then $(1 - x)(1 - y)(1 - z) = p$

$x(1 - y)(1 - z) = \alpha$

$(1 - x)\, y(1 - z) = \beta$

$(1 - x)(1 - y)(1 - z) = \gamma$

so $\dfrac{1-x}{x} = \dfrac{p}{\alpha}$ $\quad x = \dfrac{\alpha}{\alpha + p}$

similarly z = $\dfrac{\gamma}{\gamma + p}$

so $\dfrac{P(E_1)}{P(E_3)} = \dfrac{\dfrac{\alpha}{\alpha + p}}{\dfrac{\gamma}{\gamma + p}} = \dfrac{\dfrac{\gamma + p}{\gamma}}{\dfrac{\alpha + p}{\alpha}} = \dfrac{1 + \dfrac{p}{\gamma}}{1 + \dfrac{p}{\alpha}}$

also given $\dfrac{\alpha\beta}{\alpha - 2\beta} = p = \dfrac{2\beta\gamma}{\beta - 3\gamma} \Rightarrow \beta = \dfrac{5\alpha\gamma}{\alpha + 4\gamma}$

Substituting back $\left(\alpha - 2\left(\dfrac{5\alpha\gamma}{\alpha + 4\gamma}\right)\right)\text{p} = \dfrac{\alpha \cdot 5\alpha\gamma}{\alpha + 4\gamma}$

$\Rightarrow \alpha p - 6p\gamma = 5\alpha\gamma$

$$\Rightarrow \left(\frac{p}{\gamma}+1\right)=6\left(\frac{p}{\alpha}+1\right) \Rightarrow \frac{\frac{p}{\gamma}+1}{\frac{p}{\alpha}+1}=6.$$

*58. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = _______

Sol. **(6)**

Let T_{r-1}, T_r, T_{r+1} are three consecutive terms of $(1 + x)^{n+5}$

$T_{r-1} = {}^{n+5}C_{r-2}(x)^{r-2}$, $T_r = {}^{n+5}C_{r-1}x^{r-1}$, $T_{r+1} = {}^{n+5}C_r x^r$

Where, ${}^{n+5}C_{r-2} : {}^{n+5}C_{r-1} : {}^{n+5}C_r = 5 : 10 : 14$.

So $\frac{{}^{n+5}C_{r-2}}{5}=\frac{{}^{n+5}C_{r-1}}{10}=\frac{{}^{n+5}C_r}{14}$

So from $\frac{{}^{n+5}C_{r-2}}{5}=\frac{{}^{n+5}C_{r-1}}{10} \Rightarrow n-3r=-3$ ….. (1)

$\frac{{}^{n+5}C_{r-1}}{10}=\frac{{}^{n+5}C_r}{14} \Rightarrow 5n-12r=-30$ ….. (2)

From equation (1) and (2) n = 6

*59. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = ________

Sol. **(5)**

Clearly, $1 + 2 + 3 + \ldots + n - 2 \le 1224 \le 3 + 4 + \ldots n$

$\Rightarrow \frac{(n-2)(n-1)}{2} \le 1224 \le \frac{(n-2)}{2}(3+n)$

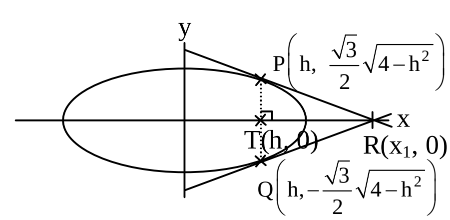
$\Rightarrow n^2 - 3n - 2446 \le 0$ and $n^2 + n - 2454 \ge 0$

$\Rightarrow 49 < n < 51 \Rightarrow n = 50$

$\therefore \frac{n(n+1)}{2}-(2k+1)=1224 \Rightarrow k = 25 \Rightarrow k - 20 = 5$

*60. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ ________

Sol. **(9)**



$\frac{x^2}{4}+\frac{y^2}{3}=1$

$y = \frac{\sqrt{3}}{2}\sqrt{4-h^2}$ at x = h

Let $R(x_1, 0)$

PQ is chord of contact, so $\frac{xx_1}{4}=1 \Rightarrow x = \frac{4}{x_1}$

which is equation of PQ, x = h

so $\frac{4}{x_1} = h \quad \Rightarrow x_1 = \frac{4}{h}$

$\Delta(h)$ = area of $\Delta PQR = \frac{1}{2} \times PQ \times RT$

$= \frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2} \times (x_1 - h) = \frac{\sqrt{3}}{2h}(4 - h^2)^{3/2}$

$\Delta'(h) = \frac{-\sqrt{3}\left(4+2h^2\right)}{2h^2}\sqrt{4-h^2}$ which is always decreasing

so Δ_1 = maximum of $\Delta(h) = \frac{45\sqrt{5}}{8}$ at $h = \frac{1}{2}$

Δ_2 = minimum of $\Delta(h) = \frac{9}{2}$ at $h = 1$

so $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = \frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8} - 8\cdot\frac{9}{2} = 45 - 36 = 9$.

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