JEE(Advanced) – 2018 TEST PAPER - 2 WITH SOLUTION

(Exam Date: 20-05-2018)

PART-1 : MATHEMATICS

SECTION 1

1. For any positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ?

(A) $\sum_{j=1}^{5} \tan^2\left(f_j(0)\right) = 55$

(B) $\sum_{j=1}^{10} \left(1+f'_j(0)\right)\sec^2\left(f_j(0)\right) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan\left(f_n(x)\right) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2\left(f_n(x)\right) = 1$

Ans. (D)

Sol. $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$

$$f_n(x) = \sum_{j=1}^{n} [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$f_n(x) = \tan^{-1}(x + n) - \tan^{-1}x$

$\therefore$ $\tan(f_n(x)) = \tan[\tan^{-1}(x + n) - \tan^{-1}x]$

$$\tan(f_n(x)) = \frac{(x+n)-x}{1+x(x+n)}$$

$$\tan(f_n(x)) = \frac{n}{1+x^2+nx}$$

$\therefore \quad \sec^2(f_n(x)) = 1 + \tan^2(f_n(x))$

$$\sec^2(f_n(x)) = 1+\left(\frac{n}{1+x^2+nx}\right)^2$$

$$\lim_{x\to\infty}\sec^2(f_n(x)) = \lim_{x\to\infty}1+\left(\frac{n}{1+x^2+nx}\right)^2 = 1$$

2. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ?

(A) The point (–2, 7) lies in E_1

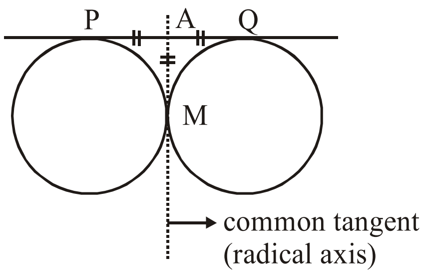
(B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2

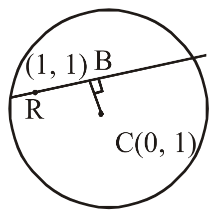
(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2

(D) The point $\left(0,\frac{3}{2}\right)$ does **NOT** lie in E_1

Ans. (D)

Sol.





AP = AQ = AM

Locus of M is a circle having PQ as its diameter

Hence, $E_1 : (x - 2)(x + 2) + (y - 7)(y + 5) = 0$ and $x \neq \pm 2$

Locus of B (midpoint)

is a circle having RC as its diameter

$E_2 : x(x - 1) + (y - 1)^2 = 0$

Now, after checking the options, we get (D)

3. Let S be the of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Ans. (A,D)

Sol. We find D = 0 & since no pair of planes are parallel, so there are infinite number of solutions.

Let $\alpha P_1 + \lambda P_2 = P_3$

$\Rightarrow \quad P_1 + 7P_2 = 13P_3$

$\Rightarrow \quad b_1 + 7b_2 = 13b_3$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) $D = 0$ but $P_1 + 7P_2 \neq 13P_3$

(C) As planes are parallel and there exist infinite ordered triplet for which they will be non coincident although satisfying $b_1 + 7b_2 = 13b_3$.

$\therefore$ rejected.

(D) $D \neq 0$

4. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

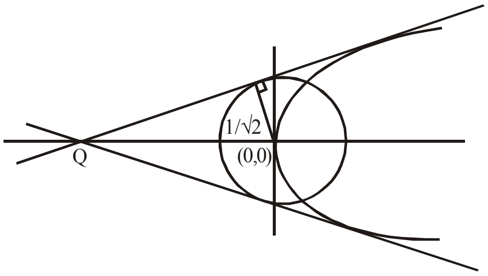
(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and x = 1 is $\frac{1}{16}(\pi - 2)$

Ans. (A,C)

Sol.



Let equation of common tangent is $y = mx + \frac{1}{m}$

$$\therefore \quad \left|\frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

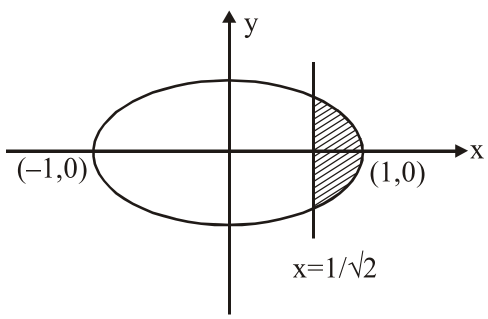
Equation of common tangents are $y = x + 1$ and $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $LR = \frac{2b^2}{a} = 1$

(C)



$$\text{Area} \quad 2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}dx = \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4}+\frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8}-\frac{1}{4}\right) = \frac{\pi-2}{4\sqrt{2}}$$

correct answer are (A) and (D)

5. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ $\left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\overline{z} + r = 0$, where $\overline{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

Ans. **(A,C,D)**

Sol. Given

$sz + t\overline{z} + r = 0$ (1)

$\overline{z} = x - iy$ (Conjugate of z)

Taking conjugate throughout $\overline{s}\overline{z} + \overline{t}z + \overline{r} = 0$ (2)

Adding (1) and (2)

$$\left(s + \overline{t}\right)z + \left(\overline{s} + t\right)\overline{z} + \left(r + \overline{r}\right) = 0$$

And Subtracting (1) and (2)

$$\left(s - \overline{t}\right)z + \left(t - \overline{s}\right)\overline{z} + \left(r - \overline{r}\right) = 0$$

For unique solution

$$\frac{t + \overline{s}}{t - s} \neq \frac{s + \overline{t}}{s - \overline{t}}$$

On further simplification $\Rightarrow |t| \neq |s|$

Hence option A proved.

If the lines coincide, then

$$\frac{t + \overline{s}}{t - \overline{s}} = \frac{\overline{t} + s}{s - \overline{t}} = \frac{r + \overline{r}}{r - \overline{r}}$$

On comparing

$$\frac{t + \overline{s}}{t - \overline{s}} = \frac{r + \overline{r}}{r - \overline{r}}$$

and simplification, we get $\Rightarrow |s| = |t|$

The lines can be parallel or coincidental.

Since, no concrete outcome.

Hence, option B is not correct.