IIT JEE Mathematics Paper 2 2007

This section contains 9 multiple choice questions numbered 1 to 9. Each question has 4 choices (A), (B), (C) and (D), out of which only one is correct.

---

1. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Which one of the following is correct?

(A) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{0}$
(B) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq\vec{0}$
(C) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{a}\times\vec{c}=\vec{0}$
(D) $\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}$ are mutually perpendicular

Sol. **(B)**

Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors and $\vec{a}+\vec{b}+\vec{c}=0$,

$\vec{a}, \vec{b}, \vec{c}$ represent an equilateral triangle.

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq 0$.

2. Let $f(x)=\frac{x}{(1+x^n)^{1/n}}$ for $n \geq 2$ and $g(x)=\underbrace{(fofo \ldots of)}_{f \text{ occurs } n \text{ times}}(x)$. Then $\int x^{n-2}g(x)dx$ equals

(A) $\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+K$
(B) $\frac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}}+K$
(C) $\frac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}}+K$
(D) $\frac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}}+K$

Sol. **(A)**

Here $ff(x)=\frac{f(x)}{[1+f(x)^n]^{1/n}}=\frac{x}{(1+2x^n)^{1/x}}$

$fff(x)=\frac{x}{(1+3x^n)^{1/n}}$

$\Rightarrow g(x)=\underbrace{(fofo...of)}_{n \text{ terms}}(x)=\frac{x}{(1+nx^n)^{1/n}}$

Hence $I=\int x^{n-2}g(x)dx=\int\frac{x^{n-1}dx}{(1+nx^n)^{1/n}}$

$=\frac{1}{n^2}\int\frac{n^2x^{n-1}dx}{(1+nx^n)^{1/n}}=\frac{1}{n^2}\int\frac{\frac{d}{dx}(1+nx^n)}{(1+nx^n)^{1/n}}dx$

$\therefore\ I=\frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}}+k$.

3. $\frac{d^2x}{dy^2}$ equals

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$
(B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$
(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$
(D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Sol. **(D)**

Since, $\frac{dx}{dy} = \frac{1}{dy/dx} = \left(\frac{dy}{dx}\right)^{-1}$

$\Rightarrow \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}\frac{dx}{dy}$

$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dx}{dy}\right) = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$.

*4. The letters of the word **COCHIN** are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word **COCHIN** is

(A) 360 (B) 192
(C) 96 (D) 48

Sol. **(C)**

COCHIN

The second place can be filled in $^4C_1$ ways and the remaining four alphabets can be arranged in 4! ways in four different places. The next 97^{th} word will be COCHIN.

Hence, there are 96 words before COCHIN.

*5. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(A) a line not passing through the origin (B) $|z| = \sqrt{2}$
(C) the x-axis (D) the y-axis

Sol. **(D)**

Let z = cosθ + sinθ, so that

$\frac{z}{1-z^2} = \frac{\cos\theta + \sin\theta}{1-(\cos 2\theta + i\sin 2\theta)}$

$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta} = \frac{\cos\theta + i\sin\theta}{-2i\sin\theta(\cos\theta + i\sin\theta)}$

$= \frac{i}{2\sin\theta}$

Hence $\frac{z}{1-z^2}$ lies on the imaginary axis i.e., x = 0.

Alternative

Let $E = \frac{z}{1-z^2} = \frac{z}{z\bar{z} - z^2} = \frac{1}{\bar{z} - z}$

which is imaginary.

*6. Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is

(A) 3 (B) 2
(C) 3/2 (D) 1

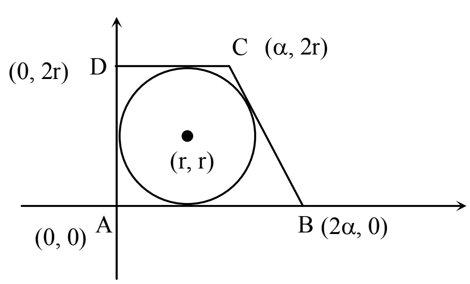
Sol. **(B)**

$18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$

Line $y = -\frac{2r}{\alpha}(x - 2\alpha)$ is tangent to $(x - r)^2 + (y - r)^2 = r^2$

$2\alpha = 3r$ and $\alpha r = 6$

r = 2.



Alternate

$\frac{1}{2}(x + 2x) \times 2r = 18$

$xr = 6$ …(1)

$\tan\theta = \frac{x - r}{r}$ $\tan(90 - \theta) = \frac{2x - r}{r}$

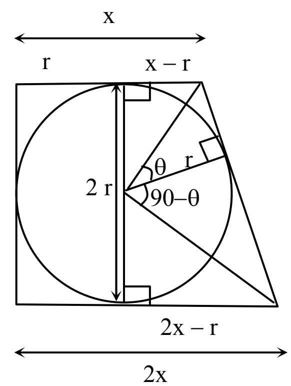
$\frac{x - r}{r} = \frac{r}{2x - r}$

$x(2x - 3r) = 0$

$x = \frac{3r}{2}$ …(2)

From (1) and (2)

r = 2.



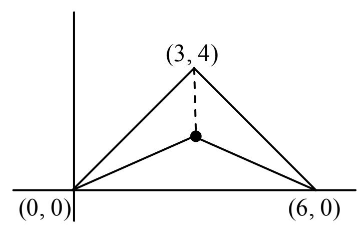
*7. Let O(0, 0), P(3, 4), Q(6, 0) be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

(A) $\left(\frac{4}{3}, 3\right)$ (B) $\left(3, \frac{2}{3}\right)$

(C) $\left(3, \frac{4}{3}\right)$ (D) $\left(\frac{4}{3}, \frac{2}{3}\right)$

Sol. **(C)**

Since, Δ is isosceles, hence centroid is the desired point.



8. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$ determines a family of circles with

(A) variable radii and a fixed centre at (0, 1)
(B) variable radii and a fixed centre at (0, –1)
(C) fixed radius 1 and variable centres along the x-axis
(D) fixed radius 1 and variable centres along the y-axis

Sol. **(C)**

$\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{y}$

$\Rightarrow \int \frac{y}{\sqrt{1 - y^2}} dy = \int dx$

$\Rightarrow -\sqrt{1 - y^2} = x + c$

$\Rightarrow (x + c)^2 + y^2 = 1$

centre (–c, 0); radius $\sqrt{c^2 - c^2 + 1} = 1$.

9. Let E^c denote the complement of an event E. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P(E^C \cap F^C \mid G)$ equals

(A) $P(E^C) + P(F^C)$ (B) $P(E^C) - P(F^C)$

(C) $P(E^C) - P(F)$ (D) $P(E) - P(F^C)$

Sol. **(C)**

$$P\left(\frac{E^c \cap F^c}{G}\right) = \frac{P(E^c \cap F^c \cap G)}{P(G)} = \frac{P(G) - P(E \cap G) - P(G \cap F)}{P(G)}$$

$= \frac{P(G)(1 - P(E) - P(F))}{P(G)}$ $[\because P(G) \neq 0]$
$= 1 - P(E) - P(F)$
$= P(E^c) - P(F)$.

SECTION –II

Assertion – Reason Type

This section contains 4 questions numbered 54 to 57. Each question contains STATEMENT – 1 (Assertion) and STATEMENT -2 (Reason). Each question has 4 choices (A), (B), (C) and (D) out of which ***ONLY ONE*** *is correct.*

10. Let $f(x) = 2 + \cos x$ for all real x.

STATEMENT -1 : For each real t, there exists a point c in $[t, t + \pi]$ such that $f'(c) = 0$.

because

STATEMENT -2 : $f(t) = f(t + 2\pi)$ for each real t.
(A) Statement -1 is True, Statement -2 is True; Statement-2 **is** a correct explanation for Statement-1
(B) Statement -1 is True, Statement -2 is True; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

Sol. **(B)**
$f(x) = 2 + \cos x \ \forall \ x \in R$
Statement : 1
There exists a point $c \in [t, t + \pi]$ where $f'(c) = 0$
Hence, statement 1 is true.
Statement 2:
$f(t) = f(t + 2\pi)$ is true.
But statement 2 is not a correct explanation for statement 1.

11. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

STATEMENT -1 : The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$

because

STATEMENT -2 : The vectors $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.
(A) Statement -1 is True, Statement -2 is True; Statement-2 **is** a correct explanation for Statement-1
(B) Statement -1 is True, Statement -2 is True; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

Sol. **(D)**
$3x - 6y - 2z = 15$
$2x + y - 2z = 5$
for $z = 0$, we get $x = 3$, $y = -1$
Direction vectors of plane are
$< 3\ \ -6\ \ -2 >$ and $<2\ \ 1\ \ -2>$
then the dr's of line of intersection of planes is $<14\ \ 2\ \ 15>$
$$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z-0}{15} = \lambda$$
$\Rightarrow x = 14\lambda + 3 \quad y = 2\lambda - 1 \quad z = 15\lambda$
Hence, statement 1 is false.
But statement 2 is true.

*12. Lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

STATEMENT -1 : The ratio PR : RQ equals $2\sqrt{2} : \sqrt{5}$.

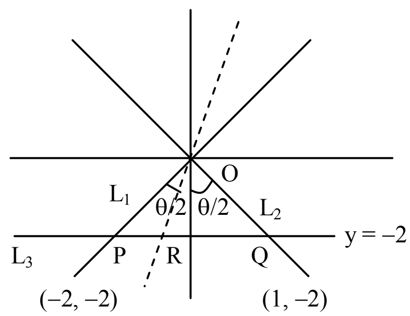
because

STATEMENT -2 : In any triangle, bisector of an angle divides the triangle into two similar triangles.
(A) Statement -1 is True, Statement -2 is true; Statement-2 **is** a correct explanation for Statement-1
(B) Statement -1 is True, Statement -2 is True; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

Sol. **(C)**

In ΔOPQ

Clearly $\frac{PR}{RQ} = \frac{OP}{OQ} = \frac{2\sqrt{2}}{\sqrt{5}}$



*13. STATEMENT -1 : The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line x = 1.

because

STATEMENT -2 : A parabola is symmetric about its axis.
(A) Statement -1 is True, Statement -2 is true; Statement-2 **is** a correct explanation for Statement-1
(B) Statement -1 is True, Statement -2 is true; Statement-2 **is NOT** a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

Sol. **(A)**

$$y = -\frac{x^2}{2} + x + 1$$

$$\Rightarrow y - \frac{3}{2} = -\frac{1}{2}(x-1)^2$$

$\Rightarrow$ it is symmetric about x = 1.

SECTION – III

Linked Comprehension Type

*This section contains 2 paragraphs M_{58-60} and M_{61-63}. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choice (A), (B), (C) and (D), out of which **ONLY ONE** is correct.*

M_{58-60} : Paragraph for question Nos. 58 to 60

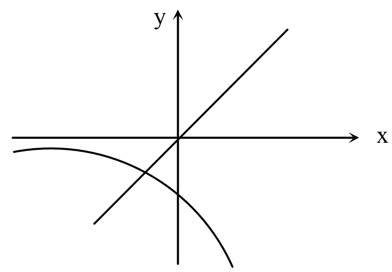
If a continuous f defined on the real line R, assumes positive and negative values in R then the equation f(x) = 0 has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum values is negative then the equation f(x) = 0 has a root in R.
Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

14. The line y = x meets $y = ke^x$ for $k \le 0$ at

(A) no point
(B) one point
(C) two points
(D) more than two points

Sol. **(B)**

Line y = x intersect the curve $y = ke^x$ at exactly one point when $k \le 0$.



15. The positive value of k for which kex - x = 0 has only one root is

(A) 1/e
(B) 1
(C) e
(D) $\log_e 2$

Sol. **(A)**

Let $f(x) = ke^x - x$

$f'(x) = ke^x - 1 = 0 \Rightarrow x = -\ln k$

$f''(x) = ke^x$

$f''(x)|_{x=-\ln k} = 1 > 0$

Hence $f(-\ln k) = 1 + \ln k$

For one root of given equation

$1 + \ln k = 0$

hence $k = \frac{1}{e}$.

16. For k > 0, the set of all values of k for which $k e^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$
(B) $\left(\frac{1}{e}, 1\right)$
(C) $\left(\frac{1}{e}, \infty\right)$
(D) (0, 1)

Sol. **(A)**

For two distinct roots $1 + \ln k < 0$ $(k > 0)$

$\ln k < -1$

$k < \frac{1}{e}$

hence $k \in \left(0, \frac{1}{e}\right)$.

M_{61-63} : Paragraph for Question Nos. 61 to 63

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \ge 2$, let A_{n-1} and H_{n-1} has arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

*17. Which one of the following statements is correct?

(A) $G_1 > G_2 > G_3 > \ldots$
(B) $G_1 < G_2 < G_3 < \ldots$
(C) $G_1 = G_2 = G_3 = \ldots$
(D) $G_1 < G_3 < G_5 < \ldots$ and $G_2 > G_4 > G_6 > \ldots$

Sol. **(C)**

$$A_1 = \frac{a+b}{2}; G_1 = \sqrt{ab}; H_1 = \frac{2ab}{a+b}$$

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}}, H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1} + H_{n-1}}$$

Clearly, $G_1 = G_2 = G_3 = \ldots = \sqrt{ab}$.

*18. Which of the following statements is correct?

(A) $A_1 > A_2 > A_3 > \ldots$ (B) $A_1 < A_2 < A_3 < \ldots$

(C) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 < \ldots$ (D) $A_1 < A_3 < A_5 < \ldots$ and $A_2 > A_4 > A_6 > \ldots$

Sol. **(A)**

A_2 is A.M. of A_1 and H_1 and $A_1 > H_1 \Rightarrow A_1 > A_2 > H_1$

A_3 is A.M. of A_2 and H_2 and $A_2 > H_2 \Rightarrow A_2 > A_3 > H_2$

$\therefore A_1 > A_2 > A_3 > \ldots$

*19. Which of the following statements is correct?

(A) $H_1 > H_2 > H_3 > \ldots$ (B) $H_1 < H_2 < H_3 < \ldots$

(C) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 < \ldots$ (D) $H_1 < H_3 < H_5 < \ldots$ and $H_2 > H_4 > H_6 > \ldots$

Sol. **(B)**

As above $A_1 > H_2 > H_1$, $A_2 > H_3 > H_2$

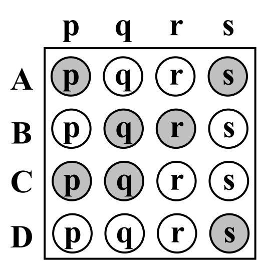
$\therefore H_1 < H_2 < H_3 < \ldots$

SECTION –IV

Matrix – Match Type

This section contains 3 questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example.

If the correct matches are A–p, A–s, B–q, B–r, C–p, C–q and D–s, then the correctly bubbled 4 × 4 matrix should be as follows:



---

20. Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$.

Match the conditions / expressions in **Column I** with statements in **Column II** and indicate your answers by darkening the appropriate bubbles in 4 × 4 matrix given in the ORS.

Column I		**Column II**
(A) If $-1 < x < 1$, then f(x) satisfies	(p)	$0 < f(x) < 1$
(B) If $1 < x < 2$, then f(x) satisfies	(q)	$f(x) < 0$
(C) If $3 < x < 5$, then f(x) satisfies	(r)	$f(x) > 0$
(D) If $x > 5$, then f(x) satisfies	(s)	$f(x) < 1$

Sol. **A → p, r, s ; B → q, s ; C → q, s ; D → p, r, s**

$f(x) = \dfrac{(x-1)(x-5)}{(x-2)(x-3)}$

The graph of f(x) is shown

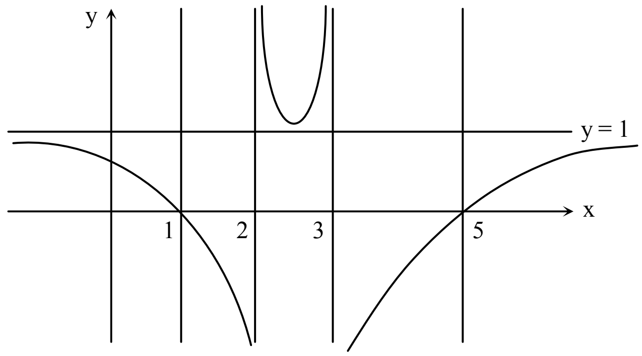
(A) If $-1 < x < 1$

$\Rightarrow 0 < f(x) < 1$

(B) If $1 < x < 2 \Rightarrow f(x) < 0$

(C) If $3 < x < 5 \Rightarrow f(x) < 0$

(D) If $x > 5 \Rightarrow 0 < f(x) < 1$



*21. Let (x, y) be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}.$$

Match the statements in **Column I** with the statements in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	**Column II**
(A) If a = 1 and b = 0, then (x, y)	(p) lies on the circle $x^2 + y^2 = 1$
(B) If a = 1 and b = 1, then (x, y)	(q) lies on $(x^2 - 1)(y^2 - 1) = 0$
(C) If a = 1 and b = 2, then (x, y)	(r) lies on $y = x$
(D) If a = 2 and b = 2, then (x, y)	(s) lies on $(4x^2 - 1)(y^2 - 1) = 0$

Sol. $\mathbf{A \to p\ ;\ B \to q\ ;\ C \to p\ ;\ D \to s}$

(A) If a = 1, b = 0

then $\sin^{-1}x + \cos^{-1}y = 0$

$\Rightarrow \sin^{-1}x = -\cos^{-1}y$

$\Rightarrow x^2 + y^2 = 1.$

(B) If a = 1 and b = 1, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}xy = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}x - \cos^{-1}y = \cos^{-1}xy$

$\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = xy$ (taking sine on both the sides)

(C) If a = 1, b = 2

$\Rightarrow \sin^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x + \cos^{-1}y = \sin^{-1}(2xy)$

$\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = 2xy$

$\Rightarrow x^2 + y^2 = 1$ (on squaring).

(D) If a = 2 and b = 2 then

$\sin^{-1}(2x) + \cos^{-1}(y) + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow 2xy + \sqrt{1-4x^2}\sqrt{1-y^2} = 2xy$

$\Rightarrow (4x^2 - 1)(y^2 - 1) = 0.$

*22. Match the statements in **Column I** with the properties **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the **ORS**.

Column I		Column II	
(A)	Two intersecting circles	(p)	have a common tangent
(B)	Two mutually external circles	(q)	have a common normal
(C)	two circles, one strictly inside the other	(r)	do not have a common tangent
(D)	two branches of a hyperbola	(s)	do not have a common normal

Sol. **A $\rightarrow$ p, q ; B $\rightarrow$ p, q ; C $\rightarrow$ q, r ; D $\rightarrow$ q, r**

(A) When two circles are intersecting they have a common normal and common tangent.
(B) Two mutually external circles have a common normal and common tangent.
(C) When one circle lies inside of other then, they have a common normal but no common tangent.
(D) Two branches of a hyperbola have a common normal but no common tangent.