JEE(Advanced) – 2017 TEST PAPER WITH SOLUTION

(HELD ON SUNDAY 21st MAY, 2017)

MATHEMATICS

SECTION–I : (Maximum Marks : 21)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –1 In all other cases

---

37. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation x + y + z = 10. Then the probability that z is even, is

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{1}{2}$

Ans. (B)

Sol. Let z = 2k, where k = 0, 1, 2, 3, 4, 5

$\therefore$ x + y = 10 – 2k

Number of non negative integral solutions

$$\sum_{k=0}^{5} {}^{11-2k}C_1 = \sum 11-2k = 36$$

Total cases = ${}^{10+3-1}C_{3-1} = 66$

Reqd. prob. = $\frac{36}{66} = \frac{6}{11}$

38. Let S = {1, 2, 3,.....,9}. For k = 1,2, ..., 5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

(A) 125 (B) 252 (C) 210 (D) 126

Ans. (D)

Sol. $N_1 + N_2 + N_3 + N_4 + N_5$ = Total ways – {when no odd}

Total ways = ${}^9C_5$

Number of ways when no odd, is zero ($\because$ only available even are 2, 4, 6, 8)

$\therefore$ **Ans :** ${}^9C_5$ – zero = 126

39. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then

(A) $0 < f'(1) \le \frac{1}{2}$ (B) $f'(1) \le 0$ (C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \le 1$

Ans. (C)

Sol. Using LMVT on $f(x)$ for $x \in \left[\frac{1}{2}, 1\right]$

$$\frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = f'(c), \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$$\frac{1 - \frac{1}{2}}{\frac{1}{2}} = f'(c) \Rightarrow f'(c) = 1, \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$\because$ $f'(x)$ is an increasing function $\forall$ x $\in$ R

$\therefore$ $f'(1) > 1$

40. If y = y(x) satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, \quad x > 0$$

and $y(0) = \sqrt{7}$, then y(256) =

(A) 80 (B) 3 (C) 16 (D) 9

Ans. (B)

Sol. $y = \frac{1}{8}\int \frac{dx}{\sqrt{4+\sqrt{9+x}}.\sqrt{x}.\sqrt{9+\sqrt{x}}}$

put $\sqrt{9+\sqrt{x}} = t \quad \Rightarrow \quad \frac{dx}{\sqrt{x}.\sqrt{9+\sqrt{x}}} = 4dt$

$\therefore \quad y = \frac{4}{8}\int \frac{dt}{\sqrt{4+t}}$

$\Rightarrow \quad y = \sqrt{4+t} + C$

$\Rightarrow \quad y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}} + C$

at x = 0 : $y(0) = \sqrt{7} \quad \Rightarrow \quad C = 0$

$\therefore \quad y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}}$

$\Rightarrow$ y(256) = 3

41. How many 3 × 3 matrices M with entries from {0,1,2} are there, for which the sum of the diagonal entries of M^TM is 5 ?

(A) 198 (B) 126 (C) 135 (D) 162

Ans. (A)

Sol. Let $M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore$ $tr(M^T M) = a^2 + b^2 + c^2 + d^2 + c^2 + f^2 + g^2 + h^2 + i^2 = 5$, where entries are $\{0,1,2\}$

Only two cases are possible.

(I) five entries 1 and other four zero

$\therefore$ ${}^9C_5 \times 1$

(II) One entry is 2, one entry is 1 and others are 0.

$\therefore$ ${}^9C_2 \times 2!$

Total = 126 + 72 = 198.

42. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

Then the triangle PQR has S as its

(A) incentre (B) orthocenter (C) circumcentre (D) centroid

Ans. **(B)**

Sol. Let position vector of $P(\vec{p})$, $Q(\vec{q})$, $R(\vec{r})$ & $S(\vec{r})$ with respect to $O(\vec{o})$

Now, $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS}$

$\Rightarrow$ $\vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{r}.\vec{p} + \vec{q}.\vec{s}$

$\Rightarrow$ $(\vec{p} - \vec{s}).(\vec{q} - \vec{r}) = 0$ .....(1)

Also, $\overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

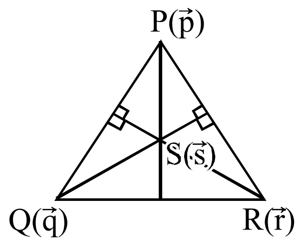
$\Rightarrow$ $\vec{r}.\vec{p} + \vec{q}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow$ $(\vec{r} - \vec{s}).(\vec{p} - \vec{q}) = 0$ ......(2)

Also, $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

$\Rightarrow$ $\vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow$ $(\vec{q} - \vec{s}).(\vec{p} - \vec{r}) = 0$ ......(3)



$\Rightarrow$ Triangle PQR has S as its orthocentre

$\therefore$ option (B) is correct.

43. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is-

(A) $14x + 2y + 15z = 31$ (B) $14x + 2y - 15z = 1$

(C) $-14x + 2y + 15z = 3$ (D) $14x - 2y + 15z = 27$

Ans. (A)

Sol. The normal vector of required plane is parallel to vector

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

$\therefore$ The equation of required plane passing through (1, 1, 1) will be

$-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$

$\Rightarrow \boxed{14x + 2y + 15z = 31}$

$\therefore$ Option (A) is correct

SECTION–2 : (Maximum Marks : 28)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in one of the following categories :

Full Marks : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened.

Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : –2 In all other cases.

- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will get +4 marks; darkening only (A) and (D) will get +2 marks; and darkening (A) and (B) will get –2 marks, as a wrong option is also darkened

---

44. If $I = \sum_{k=1}^{98} \int_{k}^{k+1} \frac{k+1}{x(x+1)} dx$, then

(A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

Ans. (B,C)

Sol. $I = \sum_{k=1}^{98}\left(\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx\right)$

$= \sum_{k=1}^{98}(k+1)\left(\int_k^{k+1}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx\right)$

$= \sum_{k=1}^{98}(k+1)\left(\left(\ell n\, x-\ell n\,(x+1)\right)_k^{k+1}\right)$

$= \sum_{k=1}^{98}(k+1)\left(\ell n\,(k+1)-\ell n\,(k+2)-\ell n\, k+\ell n\,(k+1)\right)$

$= \sum_{k=1}^{98}\left((k+1)\ell n\,(k+1)-k.\ell n\, k\right)-\sum_{k=1}^{98}\left((k+1).\ell n\,(k+2)-k.\ell n\,(k+1)\right)+\sum_{k=1}^{98}\left(\ell n\,(k+1)-\ell n\, k\right)$

(Difference series)

$\therefore$ I = (99 ℓn 99) + (–99 ℓn 100 + ℓn 2) + (ℓn 99) $= \ell n\left(\frac{2\times(99)^{100}}{(100)^{99}}\right)$ ......(1)

For option (B) :

Now, consider $(100)^{99} = (1 + 99)^{99}$

$= {}^{99}C_0 + {}^{99}C_1(99) + {}^{99}C_2(99)^2 + \ldots\ldots + {}^{99}C_{97}(99)^{97} + \underbrace{{}^{99}C_{98}(99)^{98}}_{(\text{value}=(99)^{99})} + \underbrace{{}^{99}C_{99}(99)^{99}}_{(\text{value}=(99)^{99})}$

$\Rightarrow (100)^{99} > 2.(99)^{99} \quad \Rightarrow \frac{2\times(99)^{99}}{(100)^{99}} < 1$

$\therefore \frac{2\times(99)^{100}}{(100)^{99}} < 99$ (on multiplying by 99)

$\Rightarrow I < \ell n99$

For option (C) :

Since, $\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{(x+1)^2}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)dx}{x(x+1)}$

$\Rightarrow \sum_{k=1}^{98}\left(\frac{1}{k+2}\right) < I$

(on integration)

$\Rightarrow \underbrace{\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{100}\right)}_{98\text{ terms}} < I$

$\Rightarrow \frac{98}{100} < \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{100} < I$

$\therefore I > \frac{49}{50}$

Hence option (C) is correct.

45. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then

(A) $f(x) > e^{2x}$ in $(0,\infty)$

(B) $f(x)$ is decreasing in $(0,\infty)$

(C) $f(x)$ is increasing in $(0,\infty)$

(D) $f'(x) < e^{2x}$ in $(0,\infty)$

Ans. (A,C)

Sol. Given that,

$f'(x) > 2f(x)\ \forall\ x \in R$

$\Rightarrow\ f'(x) - 2f(x) > 0\quad \forall\ x \in R$

$\therefore\ e^{-2x}\,(f'(x) - 2f(x)) > 0\ \forall\ x \in R$

$$\Rightarrow\ \frac{d}{dx}\left(e^{-2x}f(x)\right) > 0\ \forall\ x \in R$$

Let $g(x) = e^{-2x}f(x)$

Now, $g'(x) > 0\ \forall\ x \in R$

$\Rightarrow$ g(x) is strictly increasing $\forall\ x \in R$

Also, $g(0) = 1$

$\therefore\ \forall\ x > 0$

$\Rightarrow\ g(x) > g(0) = 1$

$\therefore\ e^{-2x}.f(x) > 1\ \forall\ x \in (0, \infty) \Rightarrow f(x) > e^{2x}\ \forall\ x \in (0,\infty)$

$\therefore$ option (A) is correct

As, $f'(x) > 2\ f(x) > 2e^{2x} > 2\ \forall\ x \in (0, \infty)$

$\Rightarrow$ f(x) is strictly increasing on $x \in (0, \infty)$

$\Rightarrow$ option (C) is correct

As, we have proved above that

$f'(x) > 2.e^{2x}\ \forall\ x \in (0, \infty)$

$\Rightarrow$ option (D) is incorrect

$\therefore$ options (A) and (C) are correct

46. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) f(x) attains its maximum at x = 0

(C) f(x) attains its minimum at x = 0

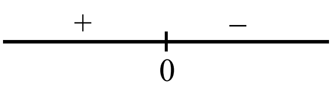
(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

Ans. (B,D)

Sol. Expansion of determinant

$f(x) = \cos 2x + \cos 4x$

$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin x(1 + 4\cos 2x)$



$\therefore$ maxima at x = 0

$f'(x) = 0 \Rightarrow x = \frac{n\pi}{2}, \cos 2x = -\frac{1}{4}$

$\Rightarrow$ more than two solutions

47. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true ?

(A) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$ (B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$ (D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

Ans. (A,C)

Sol. $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta - 1 = 0 \longrightarrow (1)$

use $\cos\beta = \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}$ and $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ in (1)

We get $\tan^2\left(\frac{\alpha}{2}\right) = 3\tan^2\left(\frac{\beta}{2}\right) \Rightarrow \tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\frac{\beta}{2} = 0$ or $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

Hence (A, C)

48. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\,dt$, then

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

Ans. (BONUS)

Sol. $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1} t\,dt \Rightarrow g'(x) = 2\sin^{-1}(\sin 2x) \times \cos 2x - \sin^{-1}(\sin x)\cos x$

$\Rightarrow g'\left(\frac{\pi}{2}\right) = 0$ & $g'\left(-\frac{\pi}{2}\right) = 0$

No option matches the result

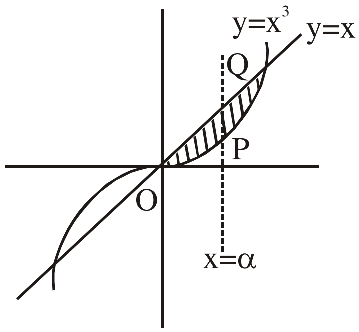
$\Rightarrow$ BONUS

49. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then

(A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \le \frac{1}{2}$ (D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

Ans. (A,D)

Sol. Area between $y = x^3$ and $y = x$ in $x \in (0,1)$ is



$$A = \int_0^1 \left(x - x^3\right) dx = \frac{1}{4}$$

Area of curve linear triangle OPQ $= \frac{A}{2} = \frac{1}{8}$

$$\Rightarrow \int_0^{\alpha} \left(x - x^3\right) dx = \frac{1}{8} \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

$$\Rightarrow \left(\alpha^2 - 1\right)^2 = \frac{1}{2} \Rightarrow \alpha^2 = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

50. Let $f(x) = \frac{1 - x\left(1 + |1 - x|\right)}{|1 - x|} \cos\left(\frac{1}{1 - x}\right)$ for $x \neq 1$. Then

(A) $\lim_{x \to 1^+} f(x)$ does not exist

(B) $\lim_{x \to 1^-} f(x)$ does not exist

(C) $\lim_{x \to 1^-} f(x) = 0$

(D) $\lim_{x \to 1^+} f(x) = 0$

Ans. (A,C)

Sol. $$f(x) = \begin{cases} (1 - x)\cos\frac{1}{1 - x} & , \quad x < 1 \\ -(1 + x)\cos\frac{1}{1 - x}, & \quad x > 1 \end{cases}$$

$$\lim_{x \to 1^+} f(x) = \text{d.n.e}, \ \lim_{x \to 1^-} f(x) = 0$$

SECTION–3 : (Maximum Marks : 12)

- This section contains **TWO** paragraphs.
- Based on each paragraph, there are **TWO** questions.
- Each question has **FOUR** options (A), (B), (C) and (D) **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 In all other cases.

PARAGRAPH 1

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

51. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$

(A) $\sin(Q + R)$ (B) $\sin(P + R)$ (C) $\sin 2R$ (D) $\sin(P + Q)$

Ans. (D)

Sol. $\overrightarrow{OX} = \frac{\overrightarrow{QR}}{QR}$

$\overrightarrow{OY} = \frac{\overrightarrow{RP}}{RP}$

$\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| = \sin R = \sin(P+Q)$

52. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

Ans. **(B)**

Sol. $-(\cos P + \cos Q + \cos R) \geq -\frac{3}{2}$ as we know cosP + cosQ + cosR will take its maximum value when

$P = Q = R = \frac{\pi}{3}$

PARAGRAPH 2

Let p,q be integers and let α,β be the roots of the equation, $x^2 - x - 1 = 0$, where α ≠ β. For n = 0,1,2,...., let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then a = 0 = b.

53. If $a_4 = 28$, then p + 2q =

(A) 14 (B) 7 (C) 12 (D) 21

Ans. **(C)**

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^4 = 3\alpha + 2$

$\therefore\ a_4 = 28 \Rightarrow p\alpha^4 + q\beta^4 = p(3\alpha + 2) + q(3\beta + 2) = 28$

$\Rightarrow\ p(3\alpha + 2) + q(3 - 3\alpha + 2) = 28$

$\Rightarrow\ \alpha(3p - 3q) + 2p + 5q = 28$ (as $\alpha \in Q^c$)

$\Rightarrow\ p = q,\ 2p + 5q = 28 \Rightarrow p = q = 4$

$\therefore\ p + 2q = 12$ **Ans : C**

54. $a_{12} =$

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

Ans. **(C)**

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\Rightarrow p\alpha^n + q\beta^n = p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$

$a_n = a_{n-1} + a_{n-2}$

$\Rightarrow a_{12} = a_{11} + a_{10}$