JEE ADVANCED (Paper - 1)

MATHEMATICS

SECTION – 1 : (One or More than One Options Correct Type)

This section contains **10 multiple choice questions**. Each question has four choices (A), (B), (C) and (D) out of which **ONE** or **MORE THAN ONE are correct.**

41. Let $f : (0, \infty) \to R$ be given by $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$, then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$
(B) $f(x)$ is monotonically decreasing on $(0, 1)$
(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$
(D) $f(2^x)$ is an odd function of x on R

42. Let $a \in R$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$, then
(A) $f(x)$ has three real roots if $a > 4$
(B) $f(x)$ has only one real roots if $a > 4$
(C) $f(x)$ has three real roots if $a < -4$
(D) $f(x)$ has three real roots if $-4 < a < 4$

43. For every pair of continuous functions $f, g : [0, 1] \to R$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is(are)
(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

*44. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then
(A) radius of S is 8
(B) radius of S is 7
(C) centre of S is (–7, 1)
(D) centre of S is (–8, 1)

45. Let $\vec{x}, \vec{y}$ *and* $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then
(A) $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$
(B) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$
(C) $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$
(D) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$

46. From a point P(λ, λ, λ), perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that ∠QPR is a right angle, then the possible value(s) of λ is(are)
(A) $\sqrt{2}$
(B) 1
(C) −1
(D) $-\sqrt{2}$

47. Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if
(A) the first column of M is the transpose of the second row of M
(B) the second row of M is the transpose of the first column of M
(C) M is a diagonal matrix with non-zero entries in the main diagonal
(D) the product of entries in the main diagonal of M is not the square of an integer

48. Let M and N be two 3×3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then
(A) determinant of $(M^2 + MN^2)$ is 0
(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(C) determinant of $(M^2 + MN^2) \geq 1$
(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

49. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & if \quad x < a, \\ \int_a^x f(t)\,dt & if \quad a \leq x \leq b, \\ \int_a^b f(t)\,dt & if \quad x > b \end{cases}$$

Then
(A) g(x) is continuous but not differentiable at a
(B) g(x) is differentiable on R
(C) g(x) is continuous but not differentiable at b
(D) g(x) is continuous and differentiable at either a or b but not both

50. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then
(A) $f(x)$ is an odd function
(B) $f(x)$ is a one-one function
(C) $f(x)$ is an onto function
(D) $f(x)$ is an even function

SECTION – 2 : (One Integer Value Correct Type)

This section contains **10 questions**. Each question, when worked out will result in one integer from 0 to 9 (both inclusive).

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*51. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is _________

*52. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is _________

53. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$$h(x) = \begin{cases} \max \ \{f(x), g(x)\} & if \quad x \leq 0 \\ \min \ \{f(x), g(x)\} & if \quad x > 0 \end{cases}.$$

Then number of points at which h(x) is not differentiable is __________

*54. Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is ________

55. Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p, q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is ______

56. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is ______

57. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} \left(1 - x^2\right)^5 \right\} dx$ is ____________

58. The largest value of the non-negative integer a for which $\lim_{x \to 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is ______

*59. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10 - x}{10}$ is ___________

*60. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is ________

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JEE(ADVANCED)-2014
PAPER 1 CODE 5
ANSWERS

MATHEMATICS

41.	**A, C, D**	42.	**B, D**	43.	**A, D**	44.	**B, C**
45.	**A, B, C**	46.	**C**	47.	**C, D**	48.	**A, B**
49.	**A, C**	50.	**A, B, C**	51.	**7**	52.	**5**
53.	**3**	54.	**4**	55.	**4**	56.	**8**
57.	**2**	58.	**2**	59.	**3**	60.	**6**

HINTS AND SOLUTIONS

MATHEMATICS

41. $f'(x) = \dfrac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$

Which is increasing in $[1, \infty)$

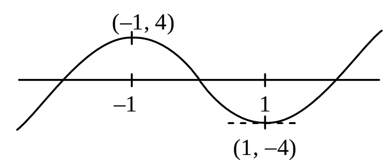
Also, $f(x) + f\left(\dfrac{1}{x}\right) = 0$

$$g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$$

$$g(-x) = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt = -g(x)$$

Hence, an odd function

42. Let $y = x^5 - 5x$



43. Let f (x) and g (x) achieve their maximum value at x_1 and x_2 respectively

$h(x) = f(x) - g(x)$

$h(x_1) = f(x_1) - g(x_1) \geq 0$

$h(x_2) = f(x_2) - g(x_2) \leq 0$

$\Rightarrow h(c) = 0$ where $c \in [0, 1] \Rightarrow f(c) = g(c)$.

44. Given circles

$x^2 + y^2 - 2x - 15 = 0$

$x^2 + y^2 - 1 = 0$

Radical axis $x + 7 = 0$ … (1)

Centre of circle lies on (1)

Let the centre be $(-7, k)$

Let equation be $x^2 + y^2 + 14x - 2ky + c = 0$

Orthogonallity gives

$-14 = c - 15 \Rightarrow c = 1$ … (2)

$(0, 1) \rightarrow 1 - 2k + 1 = 0 \Rightarrow k = 1$

Hence radius = $\sqrt{7^2 + k^2 - c} = \sqrt{49 + 1 - 1} = 7$

Alternate solution

Given circles $x^2 + y^2 - 2x - 15 = 0$

$x^2 + y^2 - 1 = 0$

Let equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Circle passes through (0, 1)

$\Rightarrow 1 + 2f + c = 0$

Applying condition of orthogonality

$-2g = c - 15,\ 0 = c - 1$

$\Rightarrow c = 1, g = 7, f = -1$

$r = \sqrt{49 + 1 - 1} = 7$; centre $(-7, 1)$

45. $\vec{a}$ is in direction of $\vec{x}\times(\vec{y}\times\vec{z})$

i.e. $(\vec{x}\cdot\vec{z})\vec{y}-(\vec{x}\cdot\vec{y})\vec{z}$

$\Rightarrow \vec{a}=\lambda_1\left[2\times\frac{1}{2}(\vec{y}-\vec{z})\right]$

$\vec{a}=\lambda_1(\vec{y}-\vec{z})$ … (1)

Now $\vec{a}\cdot\vec{y}=\lambda_1(\vec{y}\cdot\vec{y}-\vec{y}\cdot\vec{z})$

$=\lambda_1(2-1)\Rightarrow\lambda_1=\vec{a}\cdot\vec{y}$ … (2)

From (1) and (2), $\vec{a}=\vec{a}\cdot\vec{y}(\vec{y}-\vec{z})$

Similarly, $\vec{b}=(\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$

Now, $\vec{a}\cdot\vec{b}=(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})]$

$=(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})[1-1-2+1]$

$=-(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$.

46. Line 1: $\frac{x}{1}=\frac{y}{1}=\frac{z-1}{0}=r$, Q (r, r, 1)

Line 2: $\frac{x}{1}=\frac{y}{-1}=\frac{z+1}{0}=k$, R (k, – k, – 1)

$\overrightarrow{PQ}=(\lambda-r)\hat{i}+(\lambda-r)\hat{j}+(\lambda-1)\hat{k}$

and $\lambda-r+\lambda-r=0$ as $\overrightarrow{PQ}$ is $\perp$ to L_1

$\Rightarrow 2\lambda=2r\Rightarrow\lambda=r$

$\overrightarrow{PR}=(\lambda-k)\hat{i}+(\lambda+k)\hat{j}+(\lambda+1)\hat{k}$

and $\lambda-k-\lambda-k=0$ as $\overrightarrow{PR}$ is $\perp$ to L_2

$\Rightarrow k=0$

so $PQ\perp PR$

$(\lambda-r)(\lambda-k)+(\lambda-r)(\lambda+k)+(\lambda-1)(\lambda+1)=0$

$\Rightarrow\lambda=1,-1$

For $\lambda=1$ as points P and Q coincide

$\Rightarrow\lambda=-1$.

47. Let $M=\begin{bmatrix}a & c\\ c & b\end{bmatrix}$ (where $a, b, c\in I$)

then Det M = $ab-c^2$

if a = b = c, Det(M) = 0

if c = 0, a, b $\neq$ 0, Det(M) $\neq$ 0

if ab $\neq$ square of integer, Det(M) $\neq$ 0

48. $M^2=N^4\Rightarrow M^2-N^4=O\Rightarrow(M-N^2)(M+N^2)=O$ (As M, N commute.)

Also, $M\neq N^2$, Det $((M-N^2)(M+N^2))=0$

As $M-N^2$ is not null $\Rightarrow$ Det $(M+N^2)=0$

Also Det (M^2+MN^2) = (Det M) (Det $(M+N^2)$) = 0

$\Rightarrow$ There exist non-null U such that $(M^2+MN^2)U=O$

49. Since $f(x) \geq 1 \ \forall \ x \in [a, b]$
for g(x)
LHD at x = a is zero

and RHD at (x = a) = $\lim\limits_{x\to a^+} \dfrac{\int_a^x f(t)\,dt - 0}{x-a} = \lim\limits_{x\to a^+} f(x) \geq 1$

Hence g(x) is not differentiable at x = a
Similarly LHD at x = b is greater than 1
g(x) is not differentiable at x = b

50. $f(x) = (\log(\sec x + \tan x))^3 \ \forall \ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$f(-x) = -f(x)$, hence f (x) is odd function

Let $g(x) = \sec x + \tan x \ \forall \ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\Rightarrow g'(x) = \sec x(\sec x + \tan x) > 0 \ \forall \ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\Rightarrow$ g (x) is one-one function
Hence $(\log_e(g(x)))^3$ is one-one function.

and $g(x) \in (0, \infty) \ \forall \ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\Rightarrow \log(g(x)) \in R$. Hence f (x) is an onto function.

51. When n_5 takes value from 10 to 6 the carry forward moves from 0 to 4 which can be arranged in

$^4C_0 + \dfrac{^4C_1}{4} + \dfrac{^4C_2}{3} + \dfrac{^4C_3}{2} + \dfrac{^4C_4}{1} = 7$

Alternate solution
Possible solutions are
1, 2, 3, 4, 10
1, 2, 3, 5, 9
1, 2, 3, 6, 8
1, 2, 4, 5, 8
1, 2, 4, 6, 7
1, 3, 4, 5, 7
2, 3, 4, 5, 6
Hence 7 solutions are there.

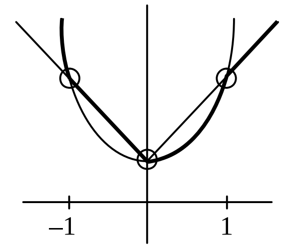
52. Number of red lines = $^nC_2 - n$
Number of blue lines = n
Hence, $^nC_2 - n = n$
$^nC_2 = 2n$

$\dfrac{n(n-1)}{2} = 2n$

$n - 1 = 4 \Rightarrow n = 5$.

53. $h(x) = \begin{cases} x^2+1 & , \quad x \in (-\infty, \ -1] \\ -x+1 & , \quad x \in [-1, \ 0] \\ x^2+1 & , \quad x \in [0, \ 1] \\ x+1 & , \quad x \in [1, \ \infty) \end{cases}$

Hence, not differentiable at x = –1, 0, 1



54. $\frac{b}{a} = \frac{c}{b} =$ (integer)

$b^2 = ac \Rightarrow c = \frac{b^2}{a}$

$\frac{a+b+c}{3} = b+2$

a + b + c = 3b + 6 ⇒ a – 2b + c = 6

$a - 2b + \frac{b^2}{a} = 6 \Rightarrow 1 - \frac{2b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$

$\left(\frac{b}{a} - 1\right)^2 = \frac{6}{a} \Rightarrow$ a = 6 only

55. $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = p + q(\vec{a} \cdot \vec{b}) + r(\vec{a} \cdot \vec{c})$

And $[\vec{a} \quad \vec{b} \quad \vec{c}] = \frac{1}{\sqrt{2}}$

$p + \frac{q}{2} + \frac{r}{2} = [\vec{a}\ \vec{b}\ \vec{c}]$ ….. (1)

$\frac{p}{2} + q + \frac{r}{2} = 0$ ….. (2)

$\frac{p}{2} + \frac{q}{2} + r = [\vec{a}\ \vec{b}\ \vec{c}]$ ….. (3)

⇒ p = r = –q

$\frac{p^2 + 2q^2 + r^2}{q^2} = 4$

56. $2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right)$

$= 1(1 + x^2)^2 + (x)(2(1 + x^2)(2x))$

Now put x = 1, y = 3 and $\frac{dy}{dx} = m$.

2 (3 – 1) (m – 5) = 1 (4) + (1) (4) (2)

$m - 5 = \frac{12}{4}$

m = 5 + 3 = 8

$\frac{dy}{dx} = m = 8$.

57. $\int_0^1 \underset{\mathrm{I}}{4x^3} \frac{d^2}{dx^2}\underset{\mathrm{II}}{\left(1-x^2\right)^5} dx$

$= \left[4x^3 \frac{d}{dx}\left(1-x^2\right)^5\right]_0^1 - \int_0^1 12x^2 \frac{d}{dx}\left(1-x^2\right)^5 dx$

$= \left[4x^3 \times 5\left(1-x^2\right)^4(-2x)\right]_0^1 - 12\left[\left[x^2\left(1-x^2\right)^5\right]_0^1 - \int_0^1 2x\left(1-x^2\right)^5 dx\right]$

$= 0 - 0 - 12[0-0] + 12\int_0^1 2x\left(1-x^2\right)^5 dx$

$= 12 \times \left[-\frac{\left(1-x^2\right)^6}{6}\right]_0^1$

$= 12\left[0+\frac{1}{6}\right] = 2$

58. $\lim_{x\to 1}\left(\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right)^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$

$\lim_{x\to 1}\left(\frac{\frac{\sin(x-1)}{(x-1)}-a}{\frac{\sin(x-1)}{(x-1)}+1}\right)^{(1+\sqrt{x})} = \frac{1}{4} \Rightarrow \left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$

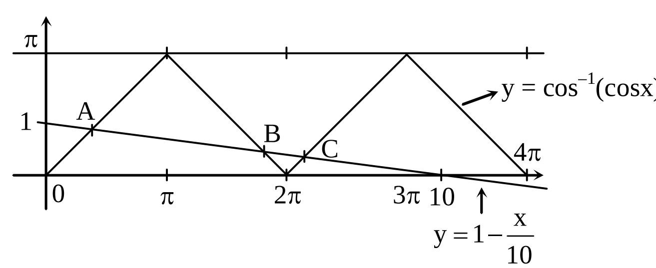
$\Rightarrow a = 0, a = 2$

$\Rightarrow a = 2$

59. $f : [0, 4\pi] \to [0, \pi]$, $f(x) = \cos^{-1}(\cos x)$

$\Rightarrow$ point A, B, C satisfy $f(x) = \frac{10-x}{10}$

Hence, 3 points



60. $2 \le d_1(p) + d_2(p) \le 4$

For $P(\alpha, \beta)$, $\alpha > \beta$

$\Rightarrow 2\sqrt{2} \le 2\alpha \le 4\sqrt{2}$

$\sqrt{2} \le \alpha \le 2\sqrt{2}$

$\Rightarrow$ Area of region $= \left(\left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2\right)$

$= 8 - 2 = 6$ sq. units