JEE ADVANCED (Paper - 1)
MATHEMATICS

SECTION 1 (Maximum Marks: 15)

- This section contains **FIVE** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories:

  Full Marks : +3 if only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 in all other cases.

  Negative Marks : –1 in all other cases.

37. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P (computer turns out to be defective given that it is produced in plant T_1)

= 10 P (computer turns out to be defective given that it is produced in plant T_2) ,

where P(E) denotes the probability of an event E. A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Key. (C)

Sol: Let $E_1 \Rightarrow$ computers produced in plant T_1

$E_2 \Rightarrow$ Computer produced in plant T_2

$A \Rightarrow$ Computer is non defective

We have to find out $P(E_2 \mid A) = \dfrac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$

Now it is given that 7% computers are defective.

Let P be the probability that computer produced in plant T_2 is defective then 10 p will be the probability that computer produced in plant T_1 is defective

So $10p \times \frac{20}{100} + p \times \frac{80}{100} = \frac{7}{100} \Rightarrow P = \frac{1}{40}$

So, $P(A \mid E_1) = 1 - \frac{10}{40} = \frac{3}{4}, P(A \mid E_2) = 1 - \frac{1}{40} = \frac{39}{40}$

$$P(E_2 \mid A) = \frac{\frac{4}{5} \times \frac{39}{40}}{\frac{4}{5} \times \frac{39}{40} + \frac{1}{5} \times \frac{3}{4}} = \frac{156}{186} = \frac{26}{31} = \frac{78}{93}$$

38. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 (B) 320 (C) 260 (D) 95

Key. (A)

Sol: Total number of ways

$$= \left(6_{C_4} \times 4_{C_0} + 6_{C_3} \times 4_{C_1}\right) \times 4_{C_1}$$

$$= (15 \times 1 + 20 \times 4) \times 4$$

= 380

39. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x \sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

Key. (C)

Sol. $x^2 - 2x\sec\theta + 1 = 0$

$$x = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$$

$$= \sec\theta \pm \tan\theta$$

$$\alpha_1 = \sec\theta - \tan\theta \left(\text{as } \alpha_1 > \beta_1\right)$$

$$x^2 + 2x\tan\theta - 1 = 0$$

$$x = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$$

$$= -\tan\theta \pm \sec\theta$$

$$\beta_2 = -\tan\theta - \sec\theta \left(\text{as } \alpha_2 > \beta_2\right)$$

So, $\alpha_1 + \beta_2 = -2\tan\theta$

40. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

Key. (C)

Sol. $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$

$$\frac{\sqrt{3}}{2}\sin x + \frac{\cos x}{2} = \cos^2 x - \sin^2 x$$

$$\cos\left(x - \frac{\pi}{3}\right) = \cos 2x$$

$$\cos 2x - \cos\left(x - \frac{\pi}{3}\right) = 0$$

$$-2\sin\left(\frac{3x - \frac{\pi}{3}}{2}\right) \times \sin\left(\frac{2x - x + \frac{\pi}{3}}{2}\right) = 0$$

(A) $\frac{3x - \frac{\pi}{3}}{2} = n\pi,$ $\qquad 3x = 2n\pi + \frac{\pi}{3}$

$x = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{-5\pi}{9}$

(B) $\frac{x + \frac{\pi}{3}}{2} = n\pi$

$x = 2n\pi - \frac{\pi}{3}$

$x = \frac{-\pi}{3}$

Sum of distinct roots from A and B = 0

41. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1,$ for all $x > 0$ is

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Key. (C)

Sol: $f(x) = 4ax^2 + \frac{1}{x}$

$f'(x) = 8ax - \frac{1}{x^2} = 0$

$x = \frac{1}{2}(a)^{-\frac{1}{3}}$

Now f" (x) = 8a + $\frac{2}{x^3}$

Which is positive for x > 0 and for positive a (as a cannot be negative otherwise f(x) ≥ 1, is not for all x > 0)

Minimum value of f(x) should be greater than or equal to 1

$$f\left(\frac{1}{2a^{\frac{1}{3}}}\right) \geq 1$$

$$4a\frac{1}{4} \times \frac{1}{a^{\frac{2}{3}}} + 2a^{\frac{1}{3}} \geq 1$$

$$3a^{\frac{1}{3}} \geq 1$$

$$a \geq \frac{1}{27}$$

SECTION 2 (Maximum Marks: 32)

- This section contains **EIGHT** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories:

  Full Marks : +4 if only the bubble(s) corresponding to all the correct option(s) is(are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, provided NO incorrect option is darkened.

  Zero Marks : 0 in all other cases.

  Negative Marks : –2 in all other cases.
- For example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks, and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened.

42. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0,\ x > 0$, passes through the point (1, 3). Then the solution curve

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at two points

(C) intersects $y = (x+2)^2$

(D) does **NOT** intersect $y = (x + 3)^2$

Sol. (A, D)

$$[(x+2)^2 + y(x+2)]\frac{dy}{dx} = y^2$$

$$y^2dx = (x + 2)^2dy + y(x + 2)dy$$

$$y[ydx - (x + 2)dy] = (x + 2)^2dy$$

$$-d\left(\frac{y}{x+2}\right) = \frac{dy}{y}$$

$$\log y = \frac{-y}{x+2} + \log k$$

This curve passes through (1, 3)

Hence, $k = 3e$

So, solution curve of given D.E. is

$$\log y = -\frac{y}{x+2} + \log 3e \qquad \text{... (i)}$$

For $y = x + 2$ (i), cut at only one point and (i) does not cut

$y = (x + 2)^2$ and $y = (x + 3)^2$

43. Consider a pyramid OPQRS located in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then

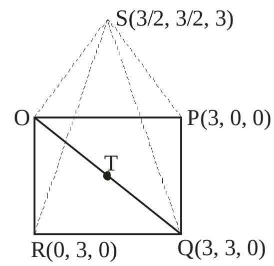
(A) the acute angle between OQ and OS is $\frac{\pi}{3}$

(B) the equation of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

Sol. (B, C, D)



$T = \left(\frac{3}{2}, \frac{3}{2}, 0\right)$, $S = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$

Let θ be the angle between OQ and OS, then

$$\tan\theta = \frac{ST}{OT} = \sqrt{2}$$

$$\theta \neq \frac{\pi}{3}$$

Equation of plane passing through O, Q and S is given by $\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 0$ and is

$x - y = 0$ ... (i)

Perpendicular distance of (i), from P is $\frac{3}{\sqrt{2}}$.

Foot of perpendicular from (0, 0, 0) to line RS is given by $\left(\frac{1}{2}, \frac{5}{2}, 1\right)$, hence perpendicular distance is $\sqrt{\frac{15}{2}}$.

44. The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Sol. (A, B, C)

For point P,

Solve parabola $x^2 = 2y$ and $x^2 + y^2 = 3$, we get value of $P(\sqrt{2}, 1)$

So equation of tangent at P is

$$\sqrt{2}x + y = 3 \quad \text{... (i)}$$

(i) also touches $x^2 + (y - k)^2 = 12$

Hence, $k = -3, 9$

Put value in $x^2 + (y - k)^2 = 12$

$\Rightarrow 3y^2 - 4ky - 6y + 2k^2 - 15 = 0$

For tangency D = 0

$k^2 - 6k - 27 = 0$

$k = -3, 9$

So, circle C_2 and C_3 are

$x^2 + (y + 3)^2 = 12$ ... (ii)

$x^2 + (y - 9)^2 = 12$ ... (iii)

$\Rightarrow Q_2 = (0, -3)$

$Q_3 = (0, 9)$

For R_2 and R_3, solving (i), (ii) and (iii), we get $R_2(2\sqrt{2}, -1)$, $R_3(-2\sqrt{2}, 7)$

45. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

Sol. (B, C)

$f : R \to R$

$f(x) = x^3 + 3x + 2$

f is invertible.

Since $g(f(x)) = x$

$\Rightarrow f(x) = g^{-1}(x)$ or $f^{-1}(x) = g(x)$

Since $g(f(x)) = x$

$\Rightarrow g'(f(x)).f^1(x) = 1$

Put x = 0

$g'(f(0)).f'(0) = 1$

$g'(2).3 = 1$

$g'(2) = \frac{1}{3}$

Also, $h(g(g(x))) = x$

$h'(g(g(x))g'(g(x))g'(x) = 1$

When x = 236

$h'(1)g'(6).g'(236) = 1$

$h'(1).\frac{1}{6}\times\frac{1}{111} = 1$

$h'(1) = 666$

and $h(g(g(x)) = x$

For x = 16 $\Rightarrow$ h(0) = 16; For x = 38 $\Rightarrow$ h(g(3)) = 38

46. Let $P=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & \alpha\\ 3 & -5 & 0\end{bmatrix}$, where $\alpha \in R$, Suppose $Q=\left[q_{ij}\right]$ is a matrix such that $PQ = kI$, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and det $\left(Q\right) = \frac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$

(B) $4\alpha - k + 8 = 0$

(C) $\det(P\,adj(Q)) = 2^9$

(D) $\det\left(Q\ adj\left(P\right)\right) = 2^{13}$

Key. (B,C)

Sol. Q= $\begin{bmatrix}q_{11} & q_{12} & q_{13}\\ q_{21} & q_{22} & q_{23}\\ q_{31} & q_{32} & q_{33}\end{bmatrix}$; $PQ = kI$

Last column comparison

$3q_{13} - q_{23} - 2q_{33} = 0 \qquad (i)$

$2q_{13} + 2q_{33} = 0 \qquad (ii)$

$3q_{13} - 5q_{23} = k \qquad (iii)$

Given $q_{23} = \frac{-k}{8}$

$\alpha = -1 \qquad (iv)$

Now given $PQ = kI$

$\Rightarrow |P||Q| = |kI|$

$(12\alpha+20)\frac{k^2}{2}=k^3$

$\Rightarrow k=4$ using EQ. (iv)

$-4-4+8=0$

(C) det $(Padj(Q)) = |P||adj\ Q|$

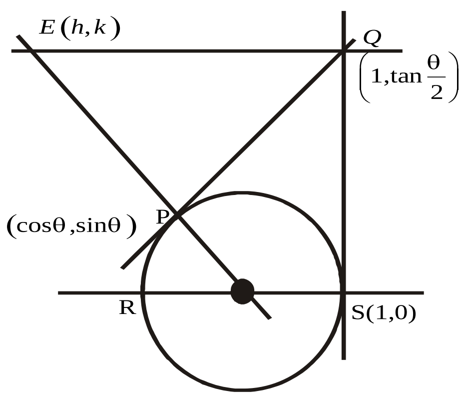
$=(12\alpha+20)|Q|^2=(8)\left(\frac{k^2}{2}\right)^2=2^3.2^6=2^9$

47. Let RS be the diameter of the circle $x^2+y^2=1$, where S is the point (1, 0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s)

(A) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4},\frac{1}{2}\right)$ (C) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{2}\right)$

Key. (A,C)

Sol.



tangent at P $x\cos\theta+y\sin\theta=1$

Tangent at 'S' is $x=1$

$\therefore$ then Q (intersection of tangent at P and S)

$\equiv\left(1,\tan\frac{\theta}{2}\right)$

Let E(h,k)

So, k = $\frac{1-\cos\theta}{\sin\theta}=\tan\frac{\theta}{2}$

K = $h\tan\theta \quad = h\left(\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right) = h\left(\frac{2k}{1-k^2}\right)$

$k(1-k^2)=2hk$

Then locus of E is $y-y^3=2xy$

Now, Option A and C are correct

48. Let $f:(0,\infty)\to R$ be a differentiable function such that $f'(x)=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. then

(A) $\lim_{x\to 0+} f'\left(\frac{1}{x}\right)=1$

(B) $\lim_{x\to 0+} xf\left(\frac{1}{x}\right)=2$

(C) $\lim_{x\to 0+} x^2f'(x)=0$

(D) $|f(x)|\leq 2$ for all $x\in(0,2)$

Key. (A)

Sol. $f'(x) = 2 - \frac{f(x)}{x}$.

$\Rightarrow xf(x) = x^2 + c \quad \Rightarrow f(x) = x + \frac{c}{x}$

As $f(1) \neq 1$ Hence $c \neq 0$.

(A) $f\left(\frac{1}{x}\right) = \frac{1}{x} + cx$

$\left(-\frac{1}{x^2}\right) f'\left(\frac{1}{x}\right) = -\frac{1}{x^2} + c \ \Rightarrow f'\left(\frac{1}{x}\right) = 1 - cx^2$

$\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = 1$ Hence A is correct

(B) $x\ f\left(\frac{1}{x}\right) = x\left(\frac{1}{x}\right) + cx^2$

$x\ f\left(\frac{1}{x}\right) = 1 + cx^2$

$\lim_{x \to 0^+} x\ f\left(\frac{1}{x}\right) = 1$ Hence B is incorrect

(C) $f'(x) = 1 - \frac{c}{x^2}$

$x^2 f'(x) = x^2 - c$

$\lim_{x \to 0^+} x^2 f'(x) = 0 - c$ But $c \neq 0$

Hence C is incorrect.

(D) Option D is not possible

49. In a triangle *XYZ*, let *x*, *y*, *z* be the lengths of sides opposite to the angles *X*, *Y*, *Z*, respectively, and $2s = x + y + z$. if $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle *XYZ* is $\frac{8\pi}{3}$, then

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Key. (A, C, D)

Sol. As $\frac{S-x}{4}=\frac{S-y}{3}=\frac{S-z}{2}$

$\pi r^2=\frac{8\pi}{3} \Rightarrow r=\frac{2\sqrt{2}}{\sqrt{3}}$ $\quad (i)$

On Simplification S = 9, x = 5, y = 6, z = 7

(A) Area of (XYZ) = $\sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$

(B) $R=\frac{abc}{4\Delta}=\frac{5.6.7}{4\times 6\sqrt{6}}=\frac{35}{4\sqrt{6}}=\frac{35\sqrt{6}}{24}$

(C) $\sin\frac{X}{2}.\sin\frac{Y}{2}.\sin\frac{Z}{2}=\frac{r}{4R}=\frac{\frac{2\sqrt{2}}{\sqrt{3}}}{4\times\frac{35}{4\sqrt{6}}}=\frac{2\sqrt{2}\times\sqrt{6}}{\sqrt{3}\times 35}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\cos^2\left(\frac{Z}{2}\right)=\frac{s(s-z)}{xy}=\frac{9(9-7)}{5\times 6}=\frac{18}{30}=\frac{3}{5}$

SECTION 3 (Maximum Marks: 15)

- This section contains FIVE questions
- The answer to each question is SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories:

  Full Marks : +3 if only the bubble corresponding to the correct answer is darkened.

  Zero Marks : 0 in all other cases

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50. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+......+(1+x)^{49}+(1+mx)^{50}$ is $(3n+1)\,{}^{51}C_3$ for same positive integer n. Then the value of n is

Key (5)

Sol. ${}^2C_2+{}^3C_2+.........+{}^{49}C_2+{}^{50}C_2m^2=(3n+1).{}^{51}C_3$

Or ${}^{50}C_3+{}^{50}C_2m^2=(3n+1)\,{}^{51}C_3$

Or $\frac{50\times 49\times 48}{3\times 2}+\frac{50\times 49}{2}m^2=(3n+1)\,{}^{51}C_3$

Or $\frac{51\times 50\times 49}{3\times 2}\left(\frac{16+m^2}{17}\right)=(3n+1)\,{}^{51}C_3$

Or ${}^{51}C_3\left(\frac{16+m^2}{17}\right)=(3n+1)\,{}^{51}C_3$

Or $\frac{16+m^2}{17}=3n+1$

Or $16+m^2=51n+17$

Or $\left(m^2-51n\right)=1$

For Smallest value of

$m=16$, $n=5$

51. Let $\alpha,\beta\in \mathbb{R}$ be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$. Then 6 $(\alpha+\beta)$ equals

Key. (7)

Sol. $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{x-\sin x}=1$ $=\lim_{x\to 0}\frac{x^2\left[\beta x-\frac{(\beta x)^3}{3!}+.....\right]}{\alpha x-\left(x-\frac{x^3}{3!}+.......\right)}=1$

$$\Rightarrow \lim_{x\to 0}\frac{\beta x^3-\frac{\beta^3 x^5}{6}+...}{(\alpha-1)x+\frac{x^3}{6}-.....}=1$$

$$\Rightarrow \alpha-1=0 \ \ and \ \ \frac{\beta}{\frac{1}{6}}=1 \ \ \Rightarrow \alpha=1,\beta=\frac{1}{6}$$

$$\alpha+\beta=\frac{7}{6} \qquad \therefore\ 6(\alpha+\beta)=7$$

52. Let $z=\frac{-1+\sqrt{3}i}{2}$ where $i=\sqrt{-1}$, and $r,s\in\{1,2,3\}$. Let $P=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r,s) for which $P^2=-I$ is

Key (1)

Sol. $z=w \qquad P=\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}$

$$P^2=-I\Rightarrow\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}\begin{bmatrix}(-z)^r & z^{2s}\\ z^{2s} & z^r\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}(-z)^{2r}+z^{4s} & (-z)^r z^{2s}+z^{2s}z^r\\ (-z)^r z^{2s}+z^r z^{2s} & z^{4s}+z^{2r}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}(-w)^{2r}+w^{4s} & (-w)^r w^{2s}+w^{2s}w^r\\ (-w)^r w^{2s}+w^r w^{2s} & w^{4s}+w^{2r}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$\Rightarrow (-w)^{2r} + w^{4s} = -1 \quad and \quad w^{4s} + w^{2r} = -1$$

$$w^{2r} + w^{s} = -1 \quad and \quad w^{s} + w^{2r} = -1$$

$$r = 1, s = 1$$

53. The total number of distinct $x \in ¡$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

Key (2)

Sol. $$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

$$\begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} + \begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} = 10$$

$$\Rightarrow x^3 \times x \times x^2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} + x \times x^2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} = 10$$

$$= x^6 2 \times 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} + x^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 3 & 6 & -2 \end{vmatrix} = 10$$

$$= 6x^6 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & 2 & 8 \end{vmatrix} + x^3(-4+6) = 10$$

$$\Rightarrow 6x^6 \times 2 \ + \ 2x^3 = 10$$

$$\Rightarrow 6x^6 + x^3 = 5$$

$x^3 = t$

Then $t = -1$, $t = 5/6$

$\Rightarrow$ $x = -1$, $x = (5/6)^{1/3}$

54. The total number of distinct $x \in [0,1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is

Ans. (1)

Sol. Let $f(x) = \int_0^x \frac{t^2}{1+t^4} dt - (2x-1)$

$$\Rightarrow f'(x) = \frac{x^2}{1+x^4} - 2$$

$$\Rightarrow \quad f'(x) < 0$$

$f(x)$ is strictly decreasing function

$$I = \frac{1}{2}\int_0^x \frac{2t^2}{t^4+1}dt = \frac{1}{2}\int_0^x \frac{(t^2+1)+(t^2-1)}{t^4+1}dt$$

$$= \frac{1}{2}\int_0^x \frac{t^2+1}{t^4+1}dt + \frac{1}{2}\int_0^x \frac{t^2-1}{t^4+1}dt$$

$$= \frac{1}{2}\int_0^x \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \frac{1}{2}\int_0^x \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$\frac{1}{2}\times\frac{1}{\sqrt{2}}\left[\tan^1\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)\right]_0^x + \frac{1}{2}\times\ \frac{1}{2\sqrt{2}}\left[\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|\right]_0^x$$

$$\frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{t^2-1}{t\sqrt{2}}\right)\right]_0^x + \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right)\right]_0^x$$

$$= \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) - \tan^{-1}(-\infty)\right] + \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}t+1}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2x}+1}\right) = 2x-1$$

$$f(0) = 1,\ f(1) < 0$$

As function is strictly decreasing, no. of root is 1.