Paper-1

JEE Advanced, 2016

Part III: Mathematics

Read the instructions carefully:

General:

1. This sealed booklet is your Question Paper. Do not break the seal till you are instructed to do so.
2. The question paper CODE is printed on the left hand top corner of this sheet and the right hand top corner of the back cover of this booklet.
3. Use the Optical Response Sheet (ORS) provided separately for answering the questions.
4. The paper CODE is printed on its left part as well as the right part of the ORS. Ensure that both these codes are identical and same as that on the question paper booklet. If not, contact the invigilator.
5. Blank spaces are provided within this booklet for rough work.
6. Write your name and roll number in the space provided on the back cover of this booklet.
7. After breaking the seal of the booklet at 9:00 am, verify that the booklet contains 36 pages and that all the 54 questions along with the options are legible. If not, contact the invigilator for replacement of the booklet.
8. You are allowed to take away the Question Paper at the end of the examination.

Optical Response Sheet

9. The ORS (top sheet) will be provided with an attached Candidate’s Sheet (bottom sheet). The Candidate’s Sheet is a carbon – less copy of the ORS.
10. Darken the appropriate bubbles on the ORS by applying sufficient pressure. This will leave an impression at the corresponding place on the Candidate’s Sheet.
11. The ORS will be collected by the invigilator at the end of the examination.
12. You will be allowed to take away the Candidate’s Sheet at the end of the examination.
13. Do not tamper with of mutilate the ORS. Do not use the ORS for rough work.

14. Write your name, roll number and code of the examination center, and sign with pen in the space provided for this purpose on the ORS. Do not write any of these details anywhere else on the ORS. Darken the appropriate bubble under each digit of your roll number.

Darken the Bubbles on the ORS

15. Use a Black Ball Point Pen to darken the bubbles on the ORS.

16. Darken the bubble ⬭ completely.

17. The correct way of darkening a bubble is as: ⬬

18. The ORS is machine – gradable. Ensure that the bubbles are darkened in the correct way.

19. Darken the bubbles only if you are sure of the answer. There is no way to erase or "un–darken" a darkened bubble.

PART - III : MATHEMATICS

SECTION–1 : (Maximum Marks : 15)

- This section contains **Five** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct
- For each question, darken the bubble corresponding to the correct option in the ORS
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –1 In all other cases

37. A debate club consists of 6 girls and 4 body. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

(A) 380 (B) 320 (C) 260 (D) 95

Ans. (A)

Sol. $({}^6C_4 + {}^6C_3.{}^4C_1).{}^4C_1 = 380$

38. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is -

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Ans. (C)

Sol. $f(x) = 4\alpha x^2 + \frac{1}{x}; x > 0$

$f'(x) = 8\alpha x - \frac{1}{x^2}$

$= \frac{8\alpha x^3 - 1}{x^2}$

$f(x)$ attains its minimum at $x = \left(\frac{1}{8\alpha}\right)^{1/3}$

$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1$

$\Rightarrow 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$

$\Rightarrow 3\alpha^{1/3} = 1 \Rightarrow \alpha = \frac{1}{27}$

39. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) 0

Ans. (C)

Sol. $\alpha_1 = \frac{2\sec\theta + \sqrt{4\sec^2\theta - 4}}{2}$ $\beta_2 = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2}$ $\{\because\ \alpha_2 > \beta_2\}$

$\alpha_1 = \sec\theta + |\tan\theta|\ \{\because\ \alpha_1 > \beta_1\}$ $\beta_2 = -\tan\theta - \sec\theta$

$\alpha_1 = \sec\theta - \tan\theta \left(\because \theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)\right)$ $\alpha_1 + \beta_2 = -2\tan\theta$

40. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to -

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

Ans. (C)

Sol. $\sqrt{3}\sin x + \cos x = 2\cos 2x$

$\Rightarrow \cos 2x = \cos\left(x - \frac{\pi}{3}\right)$

$\Rightarrow 2x = 2n\pi \pm \left(x - \frac{\pi}{3}\right)$

$x = (6n-1)\frac{\pi}{3}$ or $(6n+1)\frac{\pi}{9}$

$\Rightarrow x = -\frac{\pi}{3}, \frac{\pi}{9}, \frac{7\pi}{9}$ and $-\frac{5\pi}{9}$ in $(-\pi,\pi)$

$\Rightarrow$ sum = 0

41. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probabality that it is produced in plant T_2 is

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

Ans. (C)

Sol. $P(T_1) = \frac{20}{100}$ $\quad P(T_2) = \frac{80}{100}$

Let $P\left(\frac{D}{T_2}\right) = x \; P\left(\frac{D}{T_1}\right) = 10x \quad P(D) = \frac{7}{100}$ (given)

$$P(T_1)P\left(\frac{D}{T_1}\right) + P(T_2)P\left(\frac{D}{T_2}\right) = \frac{7}{100}$$

$$\frac{20}{100} \times 10x + \frac{80}{100} \times x = \frac{7}{100}$$

$$x = \frac{1}{40}$$

$$P\left(\frac{D}{T_2}\right) = \frac{1}{40} \Rightarrow P\left(\frac{\bar{D}}{T_2}\right) = \frac{39}{40}$$

$$P\left(\frac{D}{T_1}\right) = \frac{10}{40} \Rightarrow P\left(\frac{\bar{D}}{T_1}\right) = \frac{30}{40}$$

$$P\left(\frac{T_2}{\bar{D}}\right) = \frac{\frac{80}{100} \times \frac{39}{40}}{\frac{20}{100} \times \frac{30}{40} + \frac{80}{100} \times \frac{39}{40}} = \frac{78}{93}$$

SECTION–2 : (Maximum Marks : 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +4 If only the bubble(s) corresponding to the correct option(s) is (are) darkened.

  Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

  Zero Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –2 In all other cases.
- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened

42. A solution curve of the differential equation $(x^2+xy+4x+2y+4)\frac{dy}{dx}-y^2=0, x>0$, passes through the point (1,3). The the solution curve-

(A) intersects y = x + 2 exactly at one point

(B) intersects y = x + 2 exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersect $y = (x + 3)^2$

Ans. (A,D)

Sol. $(x^2+xy+4x+2y+4)\frac{dy}{dx}-y^2=0((x+2)^2+y(x+2))\frac{dy}{dx}=y^2$

Let x + 2 = X, y = Y $(X)(X+Y)\frac{dY}{dX}=Y^2$ $-X^2dY = XYdY - Y^2dX$

$-X^2dY = Y(XdY - YdX)$ $-\frac{dY}{Y}=\frac{XdY-YdX}{X^2}$

$$-\ell n|Y|=\left(\frac{Y}{X}\right)+C$$

$$-\ell n|y|=\frac{y}{x+2}+C$$

$\because$ it is passing through (1, 3) $-\ell n3 = 1 + C$

$C = -1 - \ell n3$

$\therefore$ curve $\frac{y}{x+2}+\ell n|y|-1-\ell n3=0, x>0$ ......(i)

put y = x + 2 in equation (i)

then $\frac{x+2}{x+2}+\ell n|x+2|-1-\ell n3=0$

x = 1, –5(reject)

$\therefore$ curve intersect y = x + 2 at point (1, 3)

for option (C), put $y = (x + 2)^2$, we will get

$x + 2 + 2\ell n(x + 2) = 1 + \ell n3$

Clearly left hand side is an increasing function Hence, it is always greater than $2 + 2\ell n2$ therefore no solution for option (C) put $y = (x + 3)^2$ in equation (i)

$$\frac{(x+3)^2}{x+2}+\ell n(x+3)^2-1-\ell n3=0$$

$$\frac{(x+3)^2}{x+2}+\ell n\frac{(x+3)^2}{3}-1=0$$

$\because \quad x > 0 \quad \Rightarrow \quad x + 3 > x + 2$

and $x + 3 > 3$

So $\frac{(x+3)^2}{x+2}+\ell n\frac{(x+3)^2}{3}>1$

$\therefore \quad \frac{(x+3)^2}{x+2}+\ell n\frac{(x+3)^2}{3}-1=0$ has no solution

$\Rightarrow$ curve $y = (x + 3)^2$ does not intersect

43. Consider a pyramid OPQRS located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP= 3. The point S is directly above the mid-point T of diagonal OQ such that TS = 3. Then-

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$

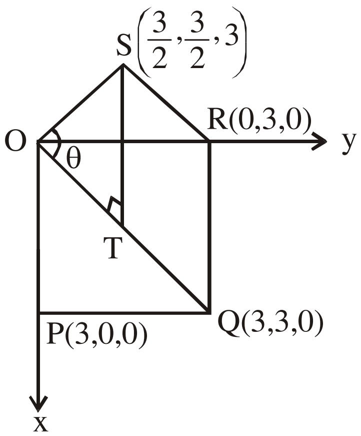
(B) the equaiton of the plane containing the triangle OQS is $x - y = 0$

(C) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

Ans. (B,C,D)

Sol.



Given OP = OR = 3 and OPQR is a square

$\Rightarrow$ $OQ = 3\sqrt{2}$ $\Rightarrow$ $OT = \frac{3}{\sqrt{2}}$ and ST = 3

using ΔSOT, $\tan\theta = \frac{ST}{OT} = \sqrt{2}$ $\Rightarrow$ $\theta = \tan^{-1}\sqrt{2}$

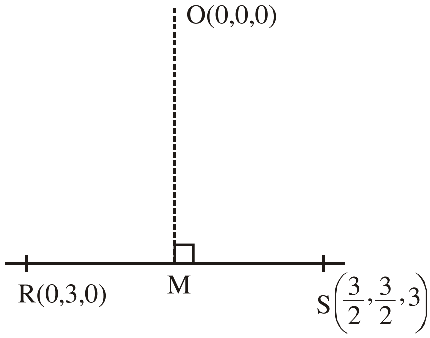
clearly, equation of plane containing triangle OQS is Y – X = 0

Also, length of perpendicular from P to the plane containing the triangle OQS is PT = $\frac{3}{\sqrt{2}}$

Also equation of RS is $\bar{r} = 3\hat{j} + t\left(\frac{3}{2}\hat{i} - \frac{3}{2}\hat{j} + 3\hat{k}\right)$

$= \left(\frac{3t}{2}, 3 - \frac{3t}{2}, 3t\right)$

Let co-ordinates of $M = \left(\frac{3t}{2}, 3 - \frac{3t}{2}, 3t\right)$



$\because$ $\overrightarrow{OM}.\overrightarrow{RS} = 0$

$\Rightarrow$ $\frac{9}{4}t - \frac{3}{2}\left(3 - \frac{3t}{2}\right) + 9t = 0$ $\Rightarrow$ $\frac{9t}{2} + 9t = \frac{9}{2}$ $\Rightarrow$ $t = \frac{1}{3}$

$\therefore$ $M = \left(\frac{1}{2}, \frac{5}{2}, 1\right)$ $\Rightarrow$ $OM = \sqrt{\frac{1}{4} + \frac{25}{4} + 1} = \sqrt{\frac{30}{4}} = \sqrt{\frac{15}{2}}$

44. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

(A) area of the triangle XYZ is $6\sqrt{6}$

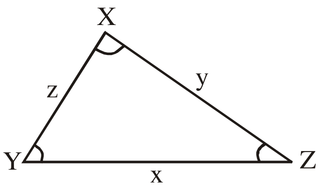
(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Ans. (A,C,D)

Sol.



Let $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}=k$

$s - x = 4k \quad \Rightarrow \quad y + z - x = 8k$

$s - y = 3k \quad \Rightarrow \quad x + z - y = 6k$

$s - z = 2k \quad \Rightarrow \quad x + y - z = 4k$

$\Rightarrow \quad x = 5k,\ y = 6k,\ z = 7k$

$\Rightarrow \quad 3s - (x + y + z) = 9k$

$s = 9k$

Let r be inradius

$\Rightarrow \quad \pi r^2 = \frac{8\pi}{3} \qquad \pi\left(\frac{\Delta}{\ }\right)^2 = \frac{8\pi}{\ } \qquad \Delta = \sqrt{\frac{8}{3}}.s$

$$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{\frac{8}{3}}.s$$

$$\sqrt{9k.4k.3k.2k} = \sqrt{\frac{8}{3}}9k$$

$$\sqrt{24.9}k^2 = \sqrt{\frac{8}{3}}.9k$$

$k = 1$

$\Rightarrow \quad x = 5,\ y = 6,\ z = 7$

$$\Delta = \sqrt{\frac{8}{3}}.9k = \sqrt{\frac{8}{3}}.9 = 6\sqrt{6}$$

$$R = \text{circumradius} = \frac{xyz}{4\Delta} = \frac{5.6.7}{4.6\sqrt{6}} = \frac{35}{4\sqrt{6}}$$

$$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{r}{4R} = \frac{\frac{\Delta}{s}}{\frac{xyz}{\Delta}} = \frac{\Delta^2}{s.xyz} = \frac{36.6}{9.5.6.7} = \frac{4}{35}$$

$$\cos Z = \frac{x^2+y^2-z^2}{2xy} = \frac{25+36-49}{2.5.6} = \frac{1}{5}$$

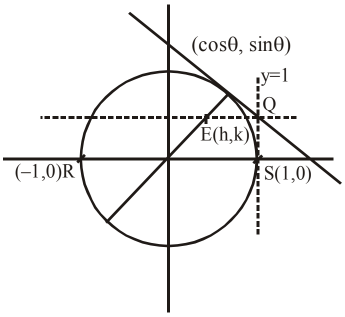
$$\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{1+\cos Z}{2} = \frac{1+\frac{1}{5}}{2} = \frac{3}{5}$$

45. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)-

(A) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{\text{[illegible]}},\frac{1}{\text{[illegible]}}\right)$ (C) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{2}\right)$

Ans. (A,C)

Sol.



Tangent at P : $x \cos\theta + y \sin\theta = 1$ ...(i)

Tangent at S : $x = 1$ ...(ii)

$\therefore$ By (i) & (ii) : $Q\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$

Line through Q parallel to RS :

$$y = \frac{1-\cos\theta}{\sin\theta} \Rightarrow y = \tan\frac{\theta}{2} \qquad \text{......(iii)}$$

Normal at P : $y = \frac{\sin\theta}{\cos\theta}x \Rightarrow y = \tan\theta.x$ ......(iv)

Point of intersection of equation (iii) and (iv), E : $h = \frac{1-\tan^2\frac{\theta}{2}}{2}; k = \tan\frac{\theta}{2}$

eliminating θ : $h = \frac{1-k^2}{2} \Rightarrow y^2 = 1-2x$

Options (A) and (C) satisfies the locus.

46. The circle $C_1: x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then-

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$

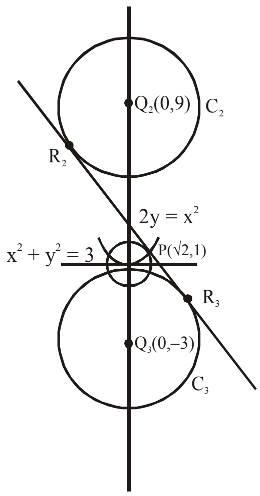
(D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Ans. (A,B,C)

Sol. On solving $x^2 + y^2 = 3$ and $x^2 = 2y$ we get point $P(\sqrt{2},1)$

Equation of tangent at P

$\sqrt{2}.x + y = 3$



Let Q_2 be (0,k) and radius is $2\sqrt{3}$

$$\therefore \left|\frac{\sqrt{2}(0)+k-3}{\sqrt{2+1}}\right| = 2\sqrt{3}$$

$\therefore$ k = 9, – 3

$Q_2(0,9)$ and $Q_3(0, -3)$

hence $Q_2Q_3 = 12$

R_2R_3 is internal common tangent of circle C_2 and C_3

$$\therefore R_2R_3 = \sqrt{(Q_2Q_3)^2 - (2\sqrt{3}+2\sqrt{3})^2}$$

$$= \sqrt{12^2 - 48} = \sqrt{96} = 4\sqrt{6}$$

Perpendicular distance of origin O from R_2R_3 is equal to radius of circle $C_1 = \sqrt{3}$

Hence area of $\Delta OR_2R_3 = \frac{1}{2}\times(R_2R_3)\sqrt{3} = \frac{1}{2}.4\sqrt{6}.\sqrt{3} = 6\sqrt{2}$

Perpendicular Distance of P from $Q_2Q_3 = \sqrt{2}$

$\therefore$ Area of $\Delta PQ_2Q_3 = \frac{1}{2}\times 12\times\sqrt{2} = 6\sqrt{2}$

47. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then-

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

Ans. (B,C)

Sol. (A) $f'(x) = 3x^2 + 3$

so, $g'(2) = \frac{1}{f'(0)}$ (Given $g(x) = f^{-1}(x)$)

$\Rightarrow \ g'(2) = \frac{1}{3}$

(B) $h(g(g(x)) = x$

$h'(g(g(x)) = \frac{1}{g'(g(x)).g'(x)}$

Now, $g(g(x)) = 1$

$g(x) = f(1) = 6$

$\therefore \ x = f(6) = 236$

so $h'(1) = \frac{1}{g'(6).g'(236)} = \frac{1}{\frac{1}{6}.\frac{1}{111}} \Rightarrow h'(1) = 666$

(C) $g(g(x)) = 0$

$\therefore \ g(x) = g^{-1}(0) \Rightarrow g(x) = f(0) \Rightarrow g(x) = 2 \Rightarrow x = g^{-1}(2) \Rightarrow x = f(2) \Rightarrow x = 16$

so $h(0) = 16$

(D) $g(x) = 3 \ \Rightarrow \ x = g^{-1}(3) \Rightarrow x = f(3) \Rightarrow x = 38$

so $h(g(3)) = 38$

48. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then

(A) $\lim\limits_{x \to 0^+} f'\left(\frac{1}{x}\right) = 1$ (B) $\lim\limits_{x \to 0^+} x f\left(\frac{1}{x}\right) = 2$ (C) $\lim\limits_{x \to 0^+} x^2 f'(x) = 0$ (D) $|f(x)| \le 2$ for all $x \in (0,2)$

Ans. (A)

Sol. Let y = f(x)

$\frac{dy}{dx}+\frac{y}{x}=2$ (linear differential equation)

$\therefore\ y.e^{\int\frac{dx}{x}}=2\int e^{\int\frac{dx}{x}}=2\int e^{\int\frac{dx}{x}}dx+c$

$\Rightarrow$ yx = $2\int xdx+c$

$\therefore$ yx = x^2 + c

$\Rightarrow f(x) = x + \frac{c}{x}$; As $f(1) \neq 1 \Rightarrow c \neq 0$

$\Rightarrow\ f'(x)=1-\frac{c}{x^2}, c\neq 0$

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}\left(1-cx^2\right)=1$

(B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+} x\left(\frac{1}{x}+cx\right)=\lim\limits_{x\to 0^+}\left(1+cx^2\right)=1$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x)=\lim\limits_{x\to 0^+} x^2\left(1-\frac{c}{x^2}\right)=\lim\limits_{x\to 0^+}\left(x^2-c\right)=-c$

(D) $f(x)=x+\frac{c}{x}, c\neq 0$

for c > 0

$\therefore\ \lim\limits_{x\to 0^+} f(x)=\infty$ $\Rightarrow$ function is not bounded in (0,2).

49. Let P $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \sim \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$, Suppose $Q = [q_{ij}]$ is a matrix such that PQ = kI, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23}=-\frac{k}{8}$ and $\det(Q)=\frac{k^2}{2}$, then-

(A) α = 0, k = 8

(B) $4\alpha - k + 8 = 0$

(C) det (Padj(Q)) = 2^9

(D) det(Qadj(P)) = 2^{13}

Ans. (B,C)

Sol. $PQ = kI$

$|P|.|Q| = k^3$

$\Rightarrow |P| = 2k \neq 0 \Rightarrow P$ is an invertible matrix

$\because PQ = kI$

$\therefore Q = kP^{-1}I$

$\therefore Q = \frac{adj.P}{2}$

$\because q_{23} = -\frac{k}{8}$

$\therefore \frac{-(3\alpha+4)}{2} = -\frac{k}{8} \Rightarrow k = 4$

$\therefore |P| = 2k \Rightarrow k = 10 + 6\alpha$ ...(i)

Put value of k in (i).. we get $\alpha = -1$

$\therefore 4\alpha - k + 8 = 0$

& $\det(P(adj.Q)) = |P|\ |adj.Q| = 2k.\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = 2^9$

SECTION–3 : (Maximum Marks : 15)

- This section contains **FIVE** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened.

Zero Marks : 0 In all other cases.

50. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is

Ans. 5

Sol. Coefficient of x^2 in the expansion of

$(1 + x)^2 + (1 + x)^3 + ...\ (1 + x)^{49} + (1 + mx)^{50}$ is

${}^2C_2 + {}^3C_2 + ...\ {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

${}^3C_3 + {}^3C_2 + ...\ {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1){}^{51}C_3$

${}^{50}C_3 + {}^{50}C_2m^2 = (3n + 1)\ {}^{51}C_3$

$\frac{50.49.48}{6} + \frac{50.49}{2}m^2 = (3n + 1)\frac{51.50.49}{6}$

$m^2 = 51n + 1$

must be a perfect quuared

$\Rightarrow$ n = 5 and m = 16

Ans. $\Rightarrow$ 5

51. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

Ans. **2**

Sol. $^{3}\begin{vmatrix} 1 & 1 & 1+x^3 \\ & & +\ ^{3} \\ 3 & 9 & 1+27x^3 \end{vmatrix} =$

$\Rightarrow \quad x^3\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} + x^3.x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 0$

$\Rightarrow \quad x^3(25 - 23) + 6x^6.2 = 10$

$\Rightarrow \quad 6x^6 + x^3 - 5 = 0$

$\Rightarrow \quad x^3 = \frac{5}{6}, -1$

two real solutions

52. Let $z = \dfrac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

Ans. **1**

Sol. $z = \omega$

$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$, $P^2 = -I$

$\Rightarrow \quad P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}((-1)^r + 1) \\ \omega^{r+2s}((-1)^r + 1) & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$

$\Rightarrow \quad (-1)^r + 1 = 0 \Rightarrow$ r is odd $\Rightarrow$ $r = 1,3$

also $\omega^{2r} + \omega^{4s} = -1 \quad \therefore \quad r \neq 3$

by $r = 1 \Rightarrow \omega^2 + \omega^{4s} = -1 \Rightarrow \quad s = 1$

$(r, s) = (1, 1)$

only 1 pair

53. The total number of distinct x ∈ [0, 1] for which $\int_0^x \frac{t^2}{1+t^4}dt$ = 2x – 1 is

Ans. 1

Sol. Let $f(x)=\int_0^x \frac{t^2}{1+t^4}dt-2x+1$

$$\Rightarrow \quad f'(x)=\frac{x^2}{1+x^4}-2$$

$$\text{as} \quad \frac{1+x^4}{x^2}\ge 2 \qquad \Rightarrow \qquad \frac{x^2}{+\quad ^4}\le\frac{1}{2}$$

$\Rightarrow \quad f'(x) \le -\frac{3}{2} \quad \Rightarrow \quad f(x)$ is continuous and decreasing

$f(0) = 1$ and $f(1)=\int_0^1 \frac{t^2}{1+t^4}dt-2\le-\frac{3}{2}$

by IVT $f(x) = 0$ possesses exactly one solution in [0, 1]

54. Let α, β ∈ R be such that $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}$ = 1. Then 6(α + β) equals

Ans. 7

Sol. If α ≠ 1, then $\lim_{x\to 0}\frac{x\sin\beta x}{\frac{\alpha x-\sin x}{x}}=0$

$$\therefore \quad \alpha=1 \quad \Rightarrow \quad \lim_{x\to 0}\frac{\beta x^3\frac{\sin\beta x}{\beta x}}{x^3\left(\frac{x-\sin x}{x^3}\right)}=\frac{\beta}{1/6}$$

$$\Rightarrow \quad 6\beta = 1 \quad \Rightarrow \quad \beta=\frac{1}{6}$$

6(α + β) = 7