JEE ADVANCED (Paper - 2)

MATHEMATICS Code - 4

Section 1 (Maximum Marks: 32)

- This section contains **EIGHT** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive.
- For each question, darken the bubble corresponding to **the** correct integer in the ORS.
- Marking scheme:
  +4 If the bubble corresponding to the answer is darkened.
  0 In all other cases.

41. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of $2x + y + z$ is

*42. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$$ is

*43. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

*44. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \ldots\ldots (1 + x^{100})$ is

*45. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. The m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m^2}+m_2^2\right)$ is

46. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$$

then the value of $\frac{m}{n}$ is

47. If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$$

where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is

48. Let f : $\mathbb{R} \to \mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\left|f\left(f(t)\right)\right|dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

Section 2 (Maximum Marks: 32)

- This section contains **EIGHT** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.
- Marking scheme:
  - +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened.
  - 0 If none of the bubbles is darkened
  - –2 In all other cases

49. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\,dx \le M$, then the possible values of m and M are

(A) $m = 13, M = 24$
(B) $m = \frac{1}{4}, M = \frac{1}{2}$
(C) $m = -11, M = 0$
(D) $m = 1, M = 12$

*50. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ?

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$
(B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$
(C) $\left(0, \frac{1}{\sqrt{5}}\right)$
(D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

*51. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

(A) $\cos\beta > 0$
(B) $\sin\beta < 0$
(C) $\cos(\alpha + \beta) > 0$
(D) $\cos\alpha < 0$

*52. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 ad E_2 at P, Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are)

(A) $e_1^2 + e_2^2 = \frac{43}{40}$
(B) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(C) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$
(D) $e_1 e_2 = \frac{\sqrt{3}}{4}$

*53. Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center N(x_2, 0). Suppose that H and S touch each other at a point P(x_1, y_1) with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

54. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt}{\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt} = L \ ?$$

(A) $a = 2$, $L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

(B) $a = 2$, $L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

(C) $a = 4$, $L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

(D) $a = 4$, $L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

55. Let $f, g : [-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

56. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)\, dx = 0$

(C) $\int_0^{\pi/4} x f(x)\, dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)\, dx = 1$

SECTION 3 (Maximum Marks: 16)

- This section contains **TWO** paragraphs.
- Based on each paragraph, there will be **TWO** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.
- Marking scheme:
  +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened.
  0 If none of the bubbles is darkened
  –2 In all other cases

PARAGRAPH 1

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

57. The correct statement(s) is(are)

(A) $f'(1) < 0$
(B) $f(2) < 0$
(C) $f'(x) \neq 0$ for any $x \in (1, 3)$
(D) $f'(x) = 0$ for some $x \in (1, 3)$

58. If $\int_1^3 x^2 F'(x)\,dx = -12$ and $\int_1^3 x^3 F''(x)\,dx = 40$, then the correct expression(s) is(are)

(A) $9f'(3) + f'(1) - 32 = 0$
(B) $\int_1^3 f(x)\,dx = 12$
(C) $9f'(3) - f'(1) + 32 = 0$
(D) $\int_1^3 f(x)\,dx = -12$

PARAGRAPH 2

Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

59. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$
(B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$
(D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

60. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are)

(A) $n_1 = 4, n_2 = 6$
(B) $n_1 = 2, n_2 = 3$
(C) $n_1 = 10, n_2 = 20$
(D) $n_1 = 3, n_2 = 6$

PAPER-2 [Code – 4]
JEE (ADVANCED) 2015
ANSWERS

MATHEMATICS

41.	**9**	42.	**4**	43.	**9**	44.	**8**
45.	**4**	46.	**2**	47.	**9**	48.	**7**
49.	**D**	50.	**A, D**	51.	**B, C, D**	52.	**A, B**
53.	**A, B, D**	54.	**A, C**	55.	**B, C**	56.	**A, B**
57.	**A, B, C**	58.	**C, D**	59.	**A, B**	60.	**C, D**

SOLUTIONS

MATHEMATICS

41. $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

$\vec{s} = x\left(-\vec{p}+\vec{q}+\vec{r}\right)+y\left(\vec{p}-\vec{q}+\vec{r}\right)+z\left(-\vec{p}-\vec{q}+\vec{r}\right)$

$\vec{s} = (-x + y - z)\,\vec{p} + (x - y - z)\,\vec{q} + (x + y + z)\,\vec{r}$

$\Rightarrow -x + y - z = 4$

$\Rightarrow x - y - z = 3$

$\Rightarrow x + y + z = 5$

On solving we get $x = 4$, $y = \frac{9}{2}$, $z = -\frac{7}{2}$

$\Rightarrow 2x + y + z = 9$

42. $$\frac{\sum_{k=1}^{12}\left|e^{i\frac{k\pi}{7}}\right|\left|e^{i\frac{\pi}{7}}-1\right|}{\sum_{k=1}^{3}\left|e^{i(4k-2)}\right|\left|e^{i\frac{\pi}{7}}-1\right|} = \frac{12}{3} = 4$$

43. Let seventh term be ‘a’ and common difference be ‘d’

Given $\frac{S_7}{S_{11}} = \frac{6}{11} \Rightarrow a = 15d$

Hence, $130 < 15d < 140$

$\Rightarrow d = 9$

44. x^9 can be formed in 8 ways

i.e. $x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+2+6}, x^{1+3+5}, x^{2+3+4}$ and coefficient in each case is 1

$\Rightarrow$ Coefficient of $x^9 = 1 + 1 + 1 + \underset{8\text{ times}}{\ldots\ldots\ldots} + 1 = 8$

45. The equation of P_1 is $y^2 - 8x = 0$ and P_2 is $y^2 + 16x = 0$

Tangent to $y^2 - 8x = 0$ passes through $(-4, 0)$

$\Rightarrow 0 = m_1(-4) + \frac{2}{m_1} \Rightarrow \frac{1}{m_1^2} = 2$

Also tangent to $y^2 + 16x = 0$ passes through $(2, 0)$

$\Rightarrow 0 = m_2 \times 2 - \frac{4}{m_2} \Rightarrow m_2^2 = 2$

$\Rightarrow \frac{1}{m_1^2} + m_2^2 = 4$

46. $$\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\frac{e}{2}$$

$$\lim_{\alpha\to 0}\frac{e\left(e^{\left(\cos(\alpha)^n - 1\right)} - 1\right)\left(\cos\alpha^n - 1\right)}{\left(\cos\left(\alpha^n\right) - 1\right)\alpha^m\alpha^{2n}}\alpha^{2n} = -\frac{e}{2} \text{ if and only if } 2n - m = 0$$

47. $\alpha = \int_0^1 e^{\left(9x+3\tan^{-1}x\right)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Put 9x + 3 $\tan^{-1}$ x = t

$\Rightarrow \left(9+\frac{3}{1+x^2}\right)dx = dt$

$\Rightarrow \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$

$\Rightarrow \left(\log_e |1+\alpha| - \frac{3\pi}{4}\right) = 9$

48. G (1) = $\int_{-1}^{1} t\left|f\left(f\left(t\right)\right)\right|dt = 0$

f (– x) = – f (x)

Given f (1) = $\frac{1}{2}$

$$\lim_{x\to 1}\frac{F(x)}{G(x)} = \lim_{x\to 1}\frac{\frac{F(x)-F(1)}{x-1}}{\frac{G(x)-G(1)}{x-1}} = \frac{f(1)}{\left|f\left(f(1)\right)\right|} = \frac{1}{14}$$

$\Rightarrow \frac{1/2}{\left|f(1/2)\right|} = \frac{1}{14}$

$\Rightarrow f\left(\frac{1}{2}\right) = 7$.

49. $\frac{192}{3}\int_{1/2}^{x} t^3 dt \le f(x) \le \frac{192}{2}\int_{1/2}^{x} t^3 dt$

$16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}$

$\int_{1/2}^{1}\left(16x^4-1\right)dx \le \int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1}\left(24x^4-\frac{3}{2}\right)dx$

$1 < \frac{26}{10} \le \int_{1/2}^{1} f(x)dx \le \frac{39}{10} < 12$

50. Here, 0 < $\left(x_1 - x_2\right)^2 < 1$

$\Rightarrow 0 < \left(x_1+x_2\right)^2 - 4x_1x_2 < 1$

$\Rightarrow 0 < \frac{1}{\alpha^2} - 4 < 1$

$\Rightarrow \alpha \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

51. $\frac{\pi}{2} < \alpha < \pi,\ \pi < \beta < \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} < \alpha + \beta < \frac{5\pi}{2}$

$\Rightarrow \sin\beta < 0; \cos\alpha < 0$

$\Rightarrow \cos(\alpha + \beta) > 0.$

52. For the given line, point of contact for $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\left(\frac{a^2}{3}, \frac{b^2}{3}\right)$

and for $E_2 : \frac{x^2}{B^2} + \frac{y^2}{A^2} = 1$ is $\left(\frac{B^2}{3}, \frac{A^2}{3}\right)$

Point of contact of x + y = 3 and circle is (1, 2)

Also, general point on x + y = 3 can be taken as $\left(1 \mp \frac{r}{\sqrt{2}}, 2 \pm \frac{r}{\sqrt{2}}\right)$ where, $r = \frac{2\sqrt{2}}{3}$

So, required points are $\left(\frac{1}{3}, \frac{8}{3}\right)$ and $\left(\frac{5}{3}, \frac{4}{3}\right)$

Comparing with points of contact of ellipse,

$a^2 = 5, B^2 = 8$

$b^2 = 4, A^2 = 1$

$\therefore\ e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$ and $e_1^2 + e_2^2 = \frac{43}{40}$

53. Tangent at P, $xx_1 - yy_1 = 1$ intersects x axis at $M\left(\frac{1}{x_1}, 0\right)$

Slope of normal = $-\frac{y_1}{x_1} = \frac{y_1 - 0}{x_1 - x_2}$

$\Rightarrow x_2 = 2x_1 \Rightarrow N \equiv (2x_1, 0)$

For centroid $\ell = \frac{3x_1 + \frac{1}{x_1}}{3}$, $m = \frac{y_1}{3}$

$\frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2}$

$\frac{dm}{dy_1} = \frac{1}{3}, \frac{dm}{dx_1} = \frac{1}{3}\frac{dy_1}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$

54. Let $\int_0^{\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt = A$

$I = \int_{\pi}^{2\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt$

Put $t = \pi + x$

$dt = dx$

for a = 2 as well as a = 4

$I = e^{\pi} \int_0^{\pi} e^x \left(\sin^6 ax + \cos^4 ax\right) dx$

$I = e^{\pi} A$

Similarly $\int_{2\pi}^{3\pi} e^t \left(\sin^6 at + \cos^4 at\right) dt = e^{2\pi} A$

So, $L = \frac{A + e^{\pi}A + e^{2\pi}A + e^{3\pi}A}{A} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$

For both a = 2, 4

55. Let H (x) = f (x) – 3g (x)

H (– 1) = H (0) = H (2) = 3.

Applying Rolle's Theorem in the interval [– 1, 0]

$H'(x) = f'(x) - 3g'(x) = 0$ for atleast one $c \in (-1, 0)$.

As $H''(x)$ never vanishes in the interval

$\Rightarrow$ Exactly one $c \in (-1, 0)$ for which $H'(x) = 0$

Similarly, apply Rolle's Theorem in the interval [0, 2].

$\Rightarrow H'(x) = 0$ has exactly one solution in (0, 2)

56. $f(x) = (7\tan^6 x - 3\tan^2 x)(\tan^2 x + 1)$

$$\int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} \left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x dx$$

$$\Rightarrow \int_0^{\pi/4} f(x)dx = 0$$

$$\int_0^{\pi/4} xf(x)dx = \left[x\int f(x)dx\right]_0^{\pi/4} - \int_0^{\pi/4}\left[\int f(x)dx\right]dx$$

$$\int_0^{\pi/4} xf(x)dx = \frac{1}{12}.$$

57. (A) $f'(x) = F(x) + xF'(x)$

$f'(1) = F(1) + F'(1)$

$f'(1) = F'(1) < 0$

$f'(1) < 0$

(B) $f(2) = 2F(2)$

F(x) is decreasing and F(1) = 0

Hence $F(2) < 0$

$\Rightarrow f(2) < 0$

(C) $f'(x) = F(x) + x F'(x)$

$F(x) < 0 \ \forall \ x \in (1, 3)$

$F'(x) < 0 \ \forall \ x \in (1, 3)$

Hence $f'(x) < 0 \ \forall \ x \in (1, 3)$

58. $$\int_1^3 f(x)dx = \int_1^3 xF(x)\, dx$$

$$= \left[\frac{x^2}{2}F(x)\right]_1^3 - \frac{1}{2}\int_1^3 x^2 F'(x)dx$$

$$= \frac{9}{2}F(3) - \frac{1}{2}F(1) + 6 \ = -12$$

$$40 = \left[x^3 F'(x)\right]_1^3 - 3\int_1^3 x^2 F'(x)\, dx$$

$40 = 27F'(3) - F'(1) + 36$ … (i)

$f'(x) = F(x) + xF'(x)$

$f'(3) = F(3) + 3F'(3)$

$f'(1) = F(1) + F'(1)$

$9f'(3) - f'(1) + 32 = 0.$

59. $P(\text{Red Ball}) = P(I)\cdot P(R \mid I) + P(II)\cdot P(R \mid II)$

$$P(II \mid R) = \frac{1}{3} = \frac{P(II)\cdot P(R \mid II)}{P(I)\cdot P(R \mid I) + P(II)\cdot P(R \mid II)}$$

$$\frac{1}{3} = \frac{\frac{n_3}{n_3 + n_4}}{\frac{n_1}{n_1 + n_2} + \frac{n_3}{n_3 + n_4}}$$

Of the given options, A and B satisfy above condition

60. P (Red after Transfer) = P(Red Transfer) . P(Red Transfer in II Case)
+ P (Black Transfer) . P(Red Transfer in II Case)

$$P(R) = \frac{n_1}{n_1 + n_2} \frac{(n_1 - 1)}{(n_1 + n_2 - 1)} + \frac{n_2}{n_1 + n_2} \cdot \frac{n_1}{n_1 + n_2 - 1} = \frac{1}{3}$$

Of the given options, option C and D satisfy above condition.