Paper-2

JEE Advanced, 2015

Part III: MATHEMATICS

Note: Answers have been highlighted in "Yellow" color and Explanations to answers are given at the end

READ THE INSTRUCTIONS CAREFULLY:

GENERAL:

1. This sealed booklet is your Question Paper. Do not break the seal till you are told to do so.

2. The question paper CODE is printed on the left hand top corner of this sheet and the right hand top corner of the back cover of this booklet.

3. Use the Optical Response Sheet (ORS) provided separately for answering the questions.

4. The ORS CODE is printed on its left part as well as the right part. Ensure that both these codes are identical and same as that on the question paper booklet. If not, contact the invigilator.

5. Blank spaces are provided within this booklet for rough work.

6. Write your name and roll number in the space provided on the back cover of this booklet.

7. After breaking the seal of the booklet, verify that the booklet contains **32** pages and that all the **60** questions along with the options are legible.

QUESTIONS PAPER FORMAT AND MARKING SCHEME:

8. The question paper has three parts: Physics, Chemistry and Mathematics, Each part has three sections.

9. Carefully read the instructions given at the beginning of each section.

10. Section 1 contains 8 questions. The answer to each question is a single digit integer ranging from 0 to 9 (both inclusive).

Marking scheme: +4 for correct answer and 0 in all other cases.

11. Section 2 contains 8 multiple choice questions with one or more than one correct option.

Marking scheme: +4 for correct answer, 0 if not attempted and -2 in all other cases.

12. Section 3 contains 2 "paragraph" type questions. Each paragraph describes an experiment, a situation or a problem. Two multiple choice questions will be asked based on this paragraph. One of or more than one option can be correct.

Marking scheme: +4 for correct answer, 0 if not attempted and -2 in all other cases.

OPTICAL RESPONSE SHEET:

13. The ORS consists of an original (top sheet) and its carbon- less copy (bottom sheet.)

14. Darken the appropriate bubbles on the original by applying sufficient pressure. This will leave an impression at the corresponding place on the carbon – less copy.

15. The original is machine – gradable and will be collected by the invigilator at the end of the examination.

16. You will be allowed to take away the carbon – less copy at the end of the examination,

17. Do not tamper with or mutilate the ORS.

18. Write your name, roll number and the name of the examination center and sign with pen in the space provided for this purpose on the original. **Do not write any of these details anywhere else**. Darken the appropriate bubble under each digit of your roll number.

<u>**Note:**</u> Answers have been highlighted in "Yellow" color and Explanations to answers are given at the end

SECTION 1(Maximum Marks: 32)

- This section contains **EIGHT** questions
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS
- Marking scheme:
  +4 If the bubble corresponding to the answer is darkened
  0 In all other cases

Q.41 The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \ldots(1 + x^{100})$ is

Ans.41 (8)

Q.42 Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2 then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

Ans.42 (4)

Q.43 Let m and n be two positive integers greater than 1. If

$$\lim_{a \to 0}\left(\frac{e^{\cos(a^n)}-e}{a^m}\right) = -\left(\frac{e}{2}\right)$$

then the value of $\frac{m}{n}$ *is*

Ans.43 (2)

Q.44 If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$$

where $\tan^{-1} x$ takes only principle values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is

Ans.44 (9)

Q.45 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous odd function which vanishes exactly at one point and $f(1) = \frac{1}{2}$.

Suppose that $F(x) = \int_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t|f(f(t))|\,dt$ for all $x \in [-1,2]$.

If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

Ans.45 (7)

Q.46 Suppose that $\vec{p}$, $\vec{q}$, and $\vec{r}$, are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p}$, $\vec{q}$ and $\vec{r}$ be 4, 3 and 5 respectively. If the components of this vector $\vec{s}$, along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of $2x+y+z$ is

Ans.46 (9)

Q.47 For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|} \textit{ is}$$

Ans.47 (4)

Q.48 Suppose that all the terms of an arithmetic progression (A. P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A. P. is

Ans.48 (9)

SECTION 2 (Maximum Marks: 32)

- This section contains **EIGHT** questions
- Each question has **Four** options (A), (b), (c) and (d). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct
- For each question, darken the bubble(s) corresponding to tall the correct option(s) in the ORS
- Marking scheme:

  +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened 0 If none of the bubbles is darkened

  0 If none of the bubbles is darkened

  –2 In all other cases

Q.49 Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

(A) $\int_0^{\pi/4} xf(x)\, dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)\, dx = 0$

(C) $\int_0^{\pi/4} xf(x)\, dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)\, dx = 1$

Ans.49 (A,B)

Q.50 Let f, g : [-1, 2] → ℝ be continuous functions which are twice differentiable on the interval.

(-1, 2). Let the values of f and g at the points – 1 , 0 and 2 be as given in the following table:

	x = –1	x = 0	x = 2
f(x)	3	6	0
g(x)	0	1	-1

In each of the intervals (-1, 0) and (0, 2) the function (f -3g)” never vanishes. Then the correct statements(s) is (are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solutions in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solutions in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

Ans.50 (B,C)

Q.51 The option(s) with the values of a and L that satisfy this following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt} = L ?$$

(A) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) $a = 4 , L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

Ans.51 (A,C)

Q.52 Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N (x_2, 0)$. Suppose that H and S touch each other at point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l,m) is the centroid of the triangle ΔPMN, then the correct expression(s) is (are)

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$.

Ans.52 (A,B,D)

Q.53 Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively.

Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are)

(A) $e_1^2 + e_2^2 = \frac{43}{40}$

(B) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

(C) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$

(D) $e_1 e_2 = \sqrt{\frac{3}{4}}$

Ans.53 (A,B,)

Q.54 If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is (are)

(A) $\cos\beta > 0$

(B) $\sin\beta < 0$.

(C) $\cos(\alpha + \beta) > 0$

(D) $\cos\alpha < 0$

Ans.54 (B,C,D)

Q.55 Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ? Which of the following intervals is (are) a subset(s) of S?

(A) $\left(-\frac{1}{2s}, -\frac{1}{\sqrt{5}}\right)$

(B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(C) $\left(0, \frac{1}{\sqrt{5}}\right)$

(D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Ans.55 (A,D)

Q.56 Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\,dx \le M$, then the possible

Values of m and M are

(A) $m = 13, M = 24$

(B) $m = \frac{1}{4}, M = \frac{1}{2}$

(C) $m = -11, M = 0$

(D) $m = 1, M = 12$

Ans.56 (D)

SECTION 3 (Maximum Marks : 16)

- This section contains **TWO** paragraphs
- Based on each paragraph, there will be **TWO** questions
- Each question ahs four options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- Marking scheme :
  +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened
  0 If none of the bubbles is darkened
  −2 In all other cases

PARAGRAPH 1

Let $F : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$.

Q.57 The correct statements(s) is(are)

(A) $f'(1) < 0$ (B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$ (D) $f'(x) = 0$ for some $x \in (1, 3)$

Ans.57 (A,B,C)

Q.58 If $\int_1^3 x^2 F'(x)\,dx = -12$ and $\int_1^3 x^3 F''(x)\,dx = 40$, then the correct expression(s) is(are)

(A) $9f'(3) + f'(1) - 32 = 0$ (B) $\int_1^3 f(x)\,dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$ (D) $\int_1^3 f(x)\,dx = -12$

Ans.58 (C,D)

PARAGRAPH 2

Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and block balls, respectively, in box II.

Q.59 One of the two boxes, box I and II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

Ans.59 (A,B)

Q.60 A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are)

(A) $n_1 = 4$ and $n_2 = 6$ (B) $n_1 = 2$ and $n_2 = 3$

(C) $n_1 = 10$ and $n_2 = 20$ (D) $n_1 = 3$ and $n_2 = 6$

Ans.60 (C,D)

Answer Keys and Explanations

Sol.41 (8)

9 = (0, 9) (1, 8), (2, 7), (3, 6), (4, 5) # 5 cases
9 = (1,2,6), (1,3,5), (2, 3, 4) # 3 cases
total = 8

Sol.42 (4)

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{5}{9} = \frac{4}{9}$$

$e = \frac{2}{3}$ focii (2, 0) (–2, 0)

$p_1 : y^2 = 8x$,

$$y = m_1 x + \frac{2}{m_1}$$

$$0 = -4m_1 + \frac{2}{m_1}$$

$\Rightarrow \quad 4m_1^2 = 2$

$\Rightarrow \quad m_1 = \pm\frac{1}{\sqrt{2}}$

$p_2 : y^2 = -16x$

$\Rightarrow \quad y = m_2 x - \frac{4}{m_2}$

$\Rightarrow \quad 0 = 2m_2 - \frac{4}{m_2}$

$\Rightarrow \quad 2m_2^2 = 4$

$$\frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$$

Sol.43 (2)

$\because \quad m \geq 2$ and $n \geq 2$

$$= \lim_{a \to 0} \frac{e(e^{\cos(a^n)-1} - 1)}{(\cos(a^n) - 1)} \times \left(\frac{\cos(a^n) - 1}{(a^n)^2}\right) \frac{a^{2n}}{a^m}$$

$$= e \times \lim_{a \to 0}\left(\frac{e^{\cos(a^n)-1} - 1)}{\cos(a^n) - 1)}\right) \times \lim_{a \to 0}\left(\frac{\cos(a^n) - 1}{a^{2n}}\right) \times \lim_{a \to 0} a^{2n-m}$$

$$= e \times 1 \times -\frac{1}{2} \times \lim_{a \to 0} a^{2n-m}$$

Now $\lim_{a \to 0} a^{2n-m}$ must be equal to 1. $\lim_{a \to 0} a^{2n-m} = 1$

i.e. $2n - m = 0$

$$\frac{m}{n} = 2$$

Sol.44 (9)

$$\alpha = \int_0^1 e^{9x-3\tan^{-1}x}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

$$\Rightarrow \quad \alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$$

$$\Rightarrow \quad \alpha = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \quad \ell n\,(1+\alpha) = 9 + \frac{3\pi}{4}$$

Alter :

$$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Let $9x + 3\tan^{-1}x = t$

$$\Rightarrow \quad \left(9 + \frac{3}{1+x^2}\right)dx = dt \quad \Rightarrow \quad \left(\frac{12+9x^2}{1+x^2}\right)dx = dt$$

$$\Rightarrow \quad \alpha = \int_0^{9+3\pi/4} e^t dt = \left(e^t\right)_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$$

Now $\log_e|1+\alpha| - 3\pi/4 = \log_e e^{(9+3\pi/4)} - 3\pi/4 = 9$

Sol.45 (7)

$$F(x) = \int_{-1}^{x} f(t)dt = \int_{1}^{x} f(t)dt$$

$$G(x) = \int_{-1}^{x} t|f(f(t))|dt = \int_{-1}^{x} t|f(f(t))|dt$$

$$\lim_{x\to1}\frac{F(x)}{G(x)}$$

L' hospitals $\lim_{x\to1}\frac{f(x)}{x\,|\,f(f(x))|} = \frac{1}{14}$

$$\frac{\frac{1}{2}}{\left|1 f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$$f\left(\frac{1}{2}\right) = 7$$

Sol.46 (9)

BONUS

This question in seem to be wrong but examiner may think like this

; g iz'u xyr yx jgk gS] jUrq ijh{kd bl izdkj lksp ldrk gS

$\vec{S} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

$\vec{S} = x(-\vec{p} + \vec{q} + \vec{r}) + y(\vec{p} - \vec{q} + \vec{r}) + z(-\vec{p} - \vec{q} + \vec{r})$

$-x + y - z = 4$ .....(1)

$x - y - z = 3$ .....(2)

$x + y + z = 5$ .....(3)

add (1) and (2)

$-2z = 7 \Rightarrow \quad z = -\frac{7}{2}$

$2x = 8 \Rightarrow \quad x = 4$

$y + z = 1$

$2x + y + z = 2(4) + 1 = 9$

Sol.47 (4)

$$\alpha_k = \cos\frac{2k\pi}{14} + i\sin\frac{2k\pi}{14} = e^{i\frac{2k\pi}{14}}$$

$$\text{Now } \frac{\sum_{k=1}^{12}\left|e^{\frac{i2(k+1)\pi}{14}} - e^{\frac{i2k\pi}{14}}\right|}{\sum_{k=1}^{3}\left|e^{\frac{i(4k-1)\pi}{14}} - e^{\frac{i(4k-2)\pi}{14}}\right|} = \frac{\sum_{k=1}^{12}\left|e^{\frac{i2\pi}{14}} - 1\right|}{\sum_{k=1}^{3}\left|e^{\frac{i2\pi}{14}} - 1\right|} = \frac{12}{3} = 4$$

Sol.48 (9)

$$\frac{S_7}{S_{11}} = \frac{6}{11}$$

$$\frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \qquad \text{Given} \qquad 130 < a + 6d < 140$$

$$\frac{7(a+3d)}{11(a+5d)} = \frac{6}{11}$$

$7a + 21d = 6a + 30d$ $\Rightarrow$ $130 < 15d < 140$

$a = 9d$

Hence $d = 9$

$a = 81$

Hence $d = 9$

Sol.49 (A, B)

$f(x) = (7\tan^6 x - 3\tan^2 x).\sec^2 x$

$$\therefore \int_0^{\pi/4} f(x)\,dx = \int_0^1 \left(7t^6 - 3t^2\right)dt = (t^7 - t^3)_0^1 = 0 \qquad \textbf{Ans. (B)}$$

$$\text{Now} \int_0^{\pi/4} xf(x)dx = \int_0^1 \underset{\text{II}}{\left(7t^6 - 3t^2\right)} \underset{\text{I}}{\tan^{-1} t}\, dt$$

$$= \left(\tan^{-1} t.\left(t^7 - t^3\right)\right)_0^1 - \int_0^1 \left(t^7 - t^3\right).\frac{1}{1+t^2}dt$$

$$= \int_0^1 \frac{t^3\left(1-t^4\right)}{1+t^2}dt = \int_0^1 t^3\left(1-t^2\right)dt$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Sol.50 (B, C)

Let $h(x) = f(x) - 3g(x)$

$\left.\begin{matrix} h(-1) = 3 \\ h(0) = 3 \end{matrix}\right\} \Rightarrow h'(x) = 0$ has atleast one root in $(-1, 0)$ and atleast one root in $(0, 2)$

$h(2) = 3$

But since $h''(x) = 0$ has no root in $(-1, 0)$ & $(0, 2)$ therefore $h'(x) = 0$ has exactly 1 root in $(-1, 0)$ & exactly 1 root in $(0, 2)$

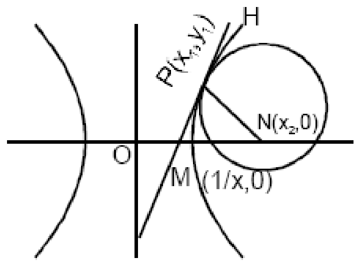
Sol.51 (A, C)

$$I_1 = \int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{\pi}^{2\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{2\pi}^{3\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt + \int_{3\pi}^{4\pi} \left(\sin^6 at + \cos^4 at\right)dt$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) \int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)dt$$

$$\Rightarrow \frac{I_1}{I_2} = 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

Sol.52 (A, B, D)



Equation tangent to H at P is $xx_1 - yy_1 = 1$

$$\ell = \frac{x_1 + x_2 + \frac{1}{x_1}}{3}, \qquad m = \frac{y_1}{3} = \frac{\sqrt{x_1^2 - 1}}{3}$$

now, $\left.\frac{dy}{dx}\right|_{H \text{ at } P} = \left.\frac{dy}{dx}\right|_{S \text{ at } P} \Rightarrow \frac{x_1}{y_1} = \frac{x_2 - x_1}{y_1} \Rightarrow x_2 = 2x_1$

So $\ell = x_1 + \frac{1}{3x_1}$

$$\frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2}, \quad \frac{dm}{dy_1} = \frac{1}{3}, \quad \frac{dm}{dx_1} = \frac{1}{3}\cdot\frac{x_1}{\sqrt{x_1^2 - 3}}$$

Sol.53 (A, B)

$E_1 \to \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$E_2 = \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

Now as $x + y = 3$ is a tangent

$a^2 + b^2 = A^2 + B^2 = 9$

Now point P is

$x^2 + (2 - x)^2 = 2$

$2x^2 - 4x + 2 = 0$

$x = 1$

so P is (1, 2)

points Q & R are $\left(\frac{5}{3}, \frac{4}{3}\right)$ & $\left(\frac{1}{3}, \frac{8}{3}\right)$

Now $\left(\frac{5}{3}, \frac{4}{3}\right)$ lies on E_1 so $\frac{25}{9a^2} + \frac{16}{9(9 - a^2)} = 1$

$\Rightarrow \quad 225 - 25a^2 + 16a^2 = 9a^2(9 - a^2) \qquad \Rightarrow \quad 225 - 9a^2 = 9a^2(9 - a^2)$

$\Rightarrow \quad 25 - a^2 = a^2(9 - a^2)$

$\Rightarrow \quad a^4 - 10a^2 + 25 = 0 \qquad \Rightarrow \quad a^2 = 5$ so $b^2 = 4$

$e_1^2 = \frac{1}{5}$

Now $\left(\frac{1}{3}, \frac{8}{3}\right)$ lies on E_2

$\frac{1}{A^2} + \frac{64}{(9 - A^2)} = 9$

$9 - A^2 + 64A^2 = 9A^2\,(9 - A^2)$

$1 + 7A^2 = A^2 = 9A^2 - A^4 \;\Rightarrow\; A^4 - 2A^2 + 1 = 0 \;\Rightarrow\; A^2 = 1$ so $B^2 = 8$

$e_2^2 = \frac{7}{8}$

Sol.54 (B, C, D)

$\alpha = 3\sin^{-1}\frac{6}{11} > 3\sin^{-1}\frac{6}{12}$ and $\beta = 3\cos^{-1}\frac{4}{9} > 3\cos^{-1}\frac{4}{8}$

$\Rightarrow \quad \alpha > \frac{\pi}{2}$ & $\beta > \pi$

$\Rightarrow \quad \alpha + \beta > \frac{3\pi}{2}$

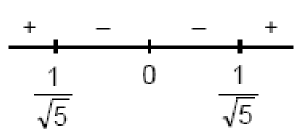
Sol.55 (A, D)

$(x_1 + x_2)^2 - 4x_1x_2 < 1$

$\frac{1}{\alpha^2} - 4 < 1$

$\Rightarrow \quad 5 - \frac{1}{\alpha^2} > 0$

$\frac{5\alpha^2 - 1}{\alpha^2} > 0$



$\alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$ ...(1)

$D > 0$

$1 - 4\alpha^2 > 0$

$\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ...(2)

(1) & (2)

$\alpha \in \left(-\frac{1}{2}, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{2}}\right)$

Sol.56 (D)

$$f'(x) = \frac{192x^3}{2+\sin^4(\pi x)} \quad \forall x \in R \; ; \; f\left(\frac{1}{2}\right) = 0$$

Now $\quad 64x^3 \le f'(x) \le 96x^3 \; \forall x \in \left[\frac{1}{2}, 1\right]$

So $\quad 16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2} \;\; \forall x \in \left[\frac{1}{2}, 1\right]$

$$\frac{16}{5}\cdot\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^{1} f(x)dx \le \frac{24}{5}\cdot\frac{31}{32} - \frac{3}{4}$$

$$\Rightarrow \quad \frac{26}{10} \le \int_{1/2}^{1} f(x)dx \le \frac{78}{20} \qquad \text{hence} \quad (D)$$

Sol.57 (A, B, C)

$f'(x) = xf'(x) + f(x)$

$\Rightarrow \quad f'(1) = f'(1) + f(1) = f'(1) < 0 \quad \Rightarrow \quad$ (A)

$f(2) = 2f(2) < 0 \quad \Rightarrow \quad$ (B)

for $x \in (1, 3) \quad f'(x) = xf'(x) + f(x) < 0 \quad \Rightarrow \quad$ (C)

Sol.58 (C, D)

$$\int_1^3 x^3 f''(x)dx = 40 \quad \Rightarrow \quad \left[x^3 f'(x)\right]_1^3 - \int_1^3 3x^2 f'(x)dx = 40$$

$\Rightarrow \quad \left[x^2 f'(x) - xf(x)\right]_1^3 - 3(-12) = 40$

$\Rightarrow \quad 9f'(3) - 3f(3) - f'(1) + f(1) = 4$

$\Rightarrow \quad 9f'(3) + 36 - f'(1) + 0 = 4 \quad \Rightarrow \quad 9f'(3) - f'(1) + 32 = 0 \quad \Rightarrow \quad$ (C)

$$\Rightarrow \quad \int_1^3 x^2 f'(x)dx = -12 \quad \Rightarrow \quad \left[x^2 f(x)\right]_1^3 - \int_1^3 2xf(x)dx = -12$$

$$\Rightarrow \quad -36 - 2\int_1^3 f(x)dx = -12 \quad \Rightarrow \quad \int_1^3 f(x)dx = -12 \quad \Rightarrow \quad (D)$$

Sol.59 (A, B)

Box – I $<$ $\begin{matrix} \text{Red} \rightarrow n_1 \\ \text{Black} \rightarrow n_2 \end{matrix}$ $\qquad$ Box – II $<$ $\begin{matrix} \text{Red} \rightarrow n_3 \\ \text{Black} \rightarrow n_4 \end{matrix}$

$$P(R) = \frac{1}{2}\cdot\frac{n_1}{n_1+n_2} + \frac{1}{2}\cdot\frac{n_3}{n_3+n_4}$$

$$R(II/R) = \frac{\frac{1}{2}.\frac{n_3}{n_3+n_4}}{\frac{1}{2}.\frac{n_1}{n_1+n_2}+\frac{1}{2}.\frac{n_3}{n_3+n_4}} = \frac{\frac{n_3}{n_3+n_4}}{\frac{n_1}{n_1+n_2}+\frac{n_3}{n_3+n_4}}$$

by option $n_1 = 3,\ n_2 = 3,\ n_3 = 5,\ n_4 = 15$

$$P(II/R) = \frac{\frac{5}{20}}{\frac{3}{6}+\frac{5}{20}} = \frac{n_4}{\frac{1}{2}+\frac{1}{4}} = \frac{1}{4}\times\frac{4}{2+1} = \frac{1}{3}$$

Sol.60 (C, D)

Given $$\frac{n_1}{n_1+n_2}.\frac{n_1-1}{n_1+n_2-1}+\frac{n_2}{n_1+n_2}.\frac{n_1}{n_1+n_2-1} = \frac{1}{3}$$

$3(n_1^2 - n_1 + n_1n_2) = (n_1 + n_2)(n_1 + n_2 - 1)$

$3n_1(n_1 + n_2 - 1) = n_1 + n_2(n_1 + n_2 - 1)$

$2n_1 = n_2$